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Convergence of a staggered Lax-Friedrichs scheme for nonlinear conservation laws on unstructured two-dimensional grids

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Summary. Based on Nessyahu and Tadmor's nonoscillatory central difference schemes for one-dimensional hyperbolic conservation laws [16], for higher dimensions several finite volume extensions and numerical results on structured and unstructured grids have been presented. The experiments show the wide applicability of these multidimensional schemes. The theoretical arguments which support this are some maximum-principles and a convergence proof in the scalar linear case. A general proof of convergence, as obtained for the original one-dimensional NT-schemes, does not exist for any of the extensions to multidimensional nonlinear problems. For the finite volume extension on two-dimensional unstructured grids introduced by Arminjon and Viallon [3, 4] we present a proof of convergence for the first order scheme in case of a nonlinear scalar hyperbolic conservation law.

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1. Introduction

The Nessyahu-Tadmor schemes, introduced in [16], are Godunov-type schemes for hyperbolic conservation laws. Their characteristic property is the use of two alternating staggered grids combined with MUSCL-type linear reconstruction and a predictor step, which yield second order accuracy. Due to the staggering the need of solving local generalized Riemann-problems at cell interfaces is bypassed. Easy application to systems of hyperbolic conservation laws is possible due to this. For these one-dimensional schemes theoretical foundation was established by the proof of convergence

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to the unique entropy solution for the scalar genuinely nonlinear case in the introductory paper of Nessyahu and Tadmor. Some modifications of the schemes have been proposed in [9,13,15].

The idea of the construction was extended to two-dimensional cartesian grids by Arminjon, Stanescu, Viallon [2]. Discrete maximum-principles for a slightly different extension were obtained by Jiang and Tadmor in the scalar case [10], by Levy and Tadmor in case of the vorticity transport equation for the $2D$–incompressible Euler-system [14].

A formulation of the NT-schemes by Arminjon, Viallon for two-dimensional unstructured grids was presented in [3,4] and convergence has been proven for the case of a linear hyperbolic equation [5]. Recently an extension of the scheme to three space dimensions has been proposed by Arminjon et al. [1].

In this paper we consider the Cauchy-problem given by the full nonlinear scalar conservation law with flux $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ and initial values u_0

$$u_t + \nabla \cdot \mathbf{f}(u) = 0 \; in \; \mathbb{R}^2 \times [0, T], \tag{1}$$

$$u(\cdot, 0) = u_0 \; in \; \mathbb{R}^2. \tag{2}$$

Our result is summed up in Theorem 4.1. We consider the basic first order scheme as proposed by Arminjon and Viallon for unstructured grids, which is the most simple NT-scheme, the staggered Lax-Friedrichs scheme.

We show that any sequence of discrete solutions defined by this staggered Lax-Friedrichs scheme converges to the unique entropy solution of the Cauchy-problem. For this we need regularity of the data, i.e. $\mathbf{f} \in C^1(\mathbb{R})^2$, $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and a non-degeneracy condition on the underlying sequence of refining space-time-grids. This is formulated by a CFL and inverse CFL-condition for the time-discretization Δt and by geometrical bounds for the finite volume cells.

To obtain this result we make use of measure-valued solutions, particularly means of DiPerna [8], which have been applied in several other convergence proofs, e.g. [11,7,12]. The plot of our proof is the following: After collecting some results concerning measure-valued solutions in Sect. 2, it just remains to show several properties of the sequence of numerical solutions in Sect. 4. First we derive an L^∞-bound for our sequence of numerical solutions. We continue with two discrete entropy inequalities, yielding uniform L^1- and L^2-stability. These inequalities are the crucial point in the proof. By an estimate of the entropy-dissipation for a special entropy-function, we obtain a kind of weak BV-estimate. This finally enables to derive estimates of weak consistency in the conservation law and the associated continuous entropy inequality, finishing the proof.

2. Convergence framework

In this section basic notations and existing results concerning measure-valued solutions are introduced. They are collected in a general convergence Theorem 2.6 at the end of the section, which will be applied to show the convergence of the staggered Lax-Friedrichs scheme. This will be formulated in Theorem 4.1.

Notation 2.1. *Let $Prob(K)$ denote the space of probability-measures on the compact set $K \subset \mathbb{R}$. For all $g \in C^0(\mathbb{R})$, $\mu \in Prob(K)$ we denote*

$$\langle \mu, g \rangle := \int_K g d\mu.$$

Definition 2.2. (Young-measure, emv-solution) *A (uniformly bounded)* **Young-measure** *is a map $\nu : \mathbb{R}^2 \times [0,T] \to Prob(K)$ for some compact $K \subset \mathbb{R}$ such that for all $g \in C^0(\mathbb{R})$ the map $\langle \nu, g \rangle(x,t) := \langle \nu(x,t), g \rangle$ is measurable.*

A Young-measure ν is an **entropy measure-valued solution** *of the conservation law (1) if*

$$\int_0^{\mathrm{T}} \int_{\mathbb{R}^2} (\langle \nu, \mathrm{id} \rangle \varphi_t + \langle \nu, \mathbf{f} \rangle \nabla \varphi) = 0, \tag{3}$$

$$\int_0^{\mathrm{T}} \int_{\mathbb{R}^2} (\langle \nu, U \rangle \varphi_t + \langle \nu, \mathbf{F} \rangle \nabla \varphi) \geq 0 \tag{4}$$

are satisfied for all entropy pairs $(U, \mathbf{F})$ (i.e. $U \in C^2(\mathbb{R})$ strictly convex, $\mathbf{F}' = U'\mathbf{f}'$) and all $\varphi \in C_0^\infty(\mathbb{R}^2 \times (0,T))$, where in (4) we additionally assume $\varphi \geq 0$. We used the notation $\langle \nu, \mathbf{f} \rangle := (\langle \nu, f_1 \rangle, \langle \nu, f_2 \rangle)^{\mathrm{T}}$.

This notion of emv-solution naturally extends the well known notion of weak entropy solution:

Remark 2.3. If an entropy measure-valued solution ν is identical to a Dirac-measure δ_u of a function $u(x,t)$ almost everywhere, then the definition exactly states that u is an entropy solution of the conservation law.

We need several results from the theory of measure-valued solutions in our proof, we list the most important ones. The first one is due to Tartar, fitted to our needs, cf. [18, 19].

Theorem 2.4. (Tartar) *Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $L^\infty(\mathbb{R}^2 \times [0,T])$ which is uniformly bounded by $B > 0$. Then a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ and a Young-measure ν exist, such that ν maps $\mathbb{R}^2 \times [0,T]$ to $Prob(K)$ with $K = [-B, B]$ and for all $g \in C^0(\mathbb{R})$ the sequence $(g(u_{k_l}))_{l\in\mathbb{N}}$ converges weak-$*$ towards $\langle \nu, g \rangle$ in $L^\infty(\mathbb{R}^2 \times [0,T])$.*

We formulate a second theorem following DiPerna [8]. It states that a Young-measure is identical to a Dirac-measure almost everywhere under certain conditions.

Theorem 2.5. (DiPerna) *Let* $f_1, f_2 \in C^1(\mathbb{R})$, $u_0 \in L^1\left(\mathbb{R}^2\right) \cap L^\infty\left(\mathbb{R}^2\right)$. *Let us further assume the existence of a Young-measure* ν *with the following properties:*

a) *the function* $\langle \nu, |\mathrm{id}| \rangle$ *is in* $L^\infty\left([0,T], L^1\left(\mathbb{R}^2\right)\right)$,
b) ν *is an entropy measure-valued solution to the conservation law (1),*
c) ν *assumes the initial values in the sense that*

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^{\mathrm{t}} \int_{\mathbb{R}^2} \langle \nu_{x,s}, |\mathrm{id} - u_0(x)| \rangle dx ds = 0.$$

Then ν *is a.e. identical to the Dirac-measure* $\delta_{u(x,t)}$ *associated to the unique weak entropy solution* $u(x,t)$ *of the Cauchy-problem (1), (2), i.e.* $\langle \nu_{x,t}, \mathrm{id} \rangle = \langle \delta_{u(x,t)}, \mathrm{id} \rangle = u(x,t)$ *almost everywhere.*

With these tools we formulate and prove a general convergence theorem. This is not a new result, it simply collects sufficient conditions which allow to apply the powerful tools mentioned above. By satisfying these conditions, one can prove convergence of any sequence of functions (not necessarily numerical solutions).

Theorem 2.6. (General convergence) *Let* $f_1, f_2 \in C^1(\mathbb{R})$, $u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), T > 0, (u_k)_{k \in \mathbb{N}}$ *be a sequence of functions in* $L^1_{loc}\left(\mathbb{R}^2 \times [0,T]\right)$, $(h_k)_{k \in \mathbb{N}}$ *a sequence of nonnegative real numbers with* $\lim_{k \to \infty} h_k = 0$.

Let nonnegative constants $C_1, C_2, \kappa, C_\varphi, C_{U,\mathbf{F},\varphi}$ *exist, such that for all* $k \in \mathbb{N}$ *the following conditions hold:*

a) $\|u_k\|_{L^\infty(\mathbb{R}^2 \times [0,T])} \le C_1$,
b) $\|u_k(\cdot,t)\|_{L^1(\mathbb{R}^2)} \le C_2$ *for all* $t \in [0,T]$,
c) *for all* $\varphi \in C_0^\infty\left(\mathbb{R}^2 \times [0,T)\right)$

$$\left| \int_0^{\mathrm{T}} \int_{\mathbb{R}^2} \left(u_k \varphi_t + \mathbf{f}(u_k) \nabla\varphi\right) + \int_{\mathbb{R}^2} u_0 \varphi(\cdot,0) \right| \le C_\varphi h_k^\kappa, \tag{5}$$

d) *for all* $\varphi \in C_0^\infty\left(\mathbb{R}^2 \times (0,T)\right)$, $\varphi \ge 0$ *and all entropy pairs* $(U, \mathbf{F})$

$$\int_0^{\mathrm{T}} \int_{\mathbb{R}^2} \left(U(u_k)\varphi_t + \mathbf{F}(u_k)\nabla\varphi\right) \ge -C_{U,\mathbf{F},\varphi} h_k^\kappa, \tag{6}$$

e) *for* $U(u) \equiv \frac{1}{2}u^2$ *and all* $t \in [0,T]$

$$\int_{\mathbb{R}^2} U\left(u_k(\cdot,t)\right) \le \int_{\mathbb{R}^2} U(u_0).$$

Then the sequence $(u_k)_{k\in\mathbb{N}}$ *converges in* $L^q_{loc}\left(\mathbb{R}^2 \times [0,T]\right)$ *for all* $1 \le q < \infty$ *strongly towards the unique entropy solution of the Cauchy-problem (1), (2).*

Proof. Because of assumption a) Thm.2.4 is applicable and we find a subsequence $(u_{k_l})_{l\in\mathbb{N}}$ and a Young-measure ν which describes weak-$*$-limits of composition-sequences $(g\left(u_{k_l}\right))_{l\in\mathbb{N}}$.

By proving the strong convergence for this subsequence, we immediately gain strong convergence of the whole starting sequence. This follows from the uniqueness of the weak entropy solution under the assumed regularities of u_0 and $\mathbf{f}$. Therefore we only deal with this subsequence and denote it as $(u_k)_{k\in\mathbb{N}}$.

Further it suffices to show that Thm.2.5 is applicable to ν. After that we know, that the values of ν are Dirac-measures $\delta_{u(x,t)}$ almost everywhere, where u denotes the unique weak entropy solution. This fact implies due to [8, Cor. 2.1] the strong convergence of the sequence $(u_k)_{k\in\mathbb{N}}$ as we claimed.

So we prove that ν satisfies conditions a) to c) of Thm.2.5.

For Thm.2.5 a) it is necessary to prove

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\mathbb{R}^2} \langle \nu, |\mathrm{id}| \rangle (\cdot, t) \le \|u_0\|_{L^1(\mathbb{R}^2)} \, .$$

This follows from assumption 2.6 b) and the weak-$*$ convergence of $(|u_k|)_{k\in\mathbb{N}}$ towards $\langle \nu, |\mathrm{id}| \rangle$, cf. [12, Thm. 7.1].

Thm.2.5 b) directly follows from assumptions 2.6 c) and d). By taking $\varphi \in C_0^\infty\left(\mathbb{R}^2 \times (0,T)\right)$, we obtain by $k \to \infty$ that the right-hand terms of (5) and (6) disappear as h_k goes to zero and the left-hand sides also converge. We directly end up with the inequalities (3), (4), stating that ν is an entropy measure-valued solution.

Condition 2.5 c) can be proven identical to e.g. [11, 7].

3. Scheme

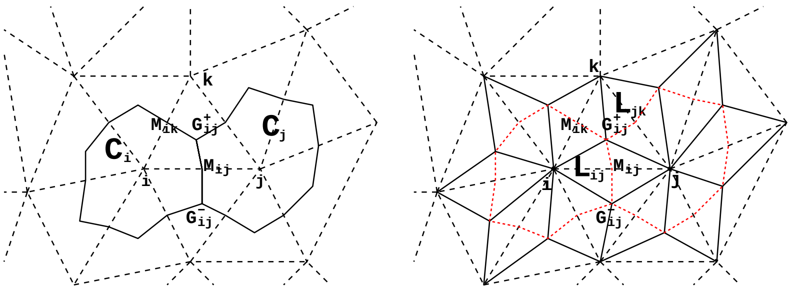
We introduce some notations needed for the definition of the scheme's finite volume cells. For illustration consider the figures below.

Notation 3.1. *Let* $\mathcal{T}$ *be a conform unstructured triangulation of* $\mathbb{R}^2$*, that is a partition in triangles, pairwise intersections of which are either empty, a common vertex or a common edge.*

- I *denotes the set of vertices in* $\mathcal{T}$*,*
- N_i *for* $i \in I$ *denotes the set of* i*'s neighbouring vertices,*
- M_{ij} *denotes the midpoint of the edge in* $\mathcal{T}$ *which connects* $i, j \in I$*,*

- $G_{ij}^+, (G_{ij}^-)$ *denotes the center of gravity of the unique triangle in* $\mathcal{T}$ *which has vertices* $i, j, k \in I$ *and for which this enumeration is positively (negatively) oriented,*
- C_i *for* $i \in I$ *denotes the dual cell around* i*, that is the polygonal area with boundary* $\bigcup_{j \in N_i} \overline{G_{ij}^+ M_{ij}} \cup \overline{G_{ij}^- M_{ij}}$*,*
- L_{ij} *for* $i \in I, j \in N_i$ *denotes the quadrangular area given by the convex hull of the vertices* i, G_{ij}^-, j, G_{ij}^+*.*

These cells C_i are taken as finite volume cells for the first step of the scheme, the cells L_{ij} are taken for the second step.



FV cells C_i for first step FV cells L_{ij} for second step

Fig. 1.

We need some further notations where $A(P)$ denotes the area of the Polygon $P \subset \mathbb{R}^2$.

Notation 3.2. *For all* $i \in I, j \in N_i$ *we define*

- $\boldsymbol{\eta}_{ij}^+, (\boldsymbol{\eta}_{ij}^-) :=$ *outer scaled normal of* C_i *for the edge* $\overline{M_{ij} G_{ij}^+}$*,* $\left(\overline{M_{ij} G_{ij}^-}\right)$*, (that means* $\left|\boldsymbol{\eta}_{ij}^+\right| = \left|\overline{M_{ij} G_{ij}^+}\right|$*),*
- $\boldsymbol{\theta}_{ij} := \boldsymbol{\eta}_{ij}^+ + \boldsymbol{\eta}_{ij}^-$*,*
- $r_{ij} := \frac{A(L_{ij} \cap C_i)}{A(C_i)}$.

Remark 3.3. With these notations for all $i \in I, j \in N_i$ hold

$$
\begin{aligned}
&M_{ij} = M_{ji}, \quad G_{ij}^+ = G_{ji}^-, \quad j \in N_i \iff i \in N_j, \\
&A(L_{ij} \cap C_i) = A(L_{ij} \cap C_j) = A(L_{ij})/2, \\
&L_{ij} = L_{ji}, \quad \boldsymbol{\eta}_{ij}^+ = -\boldsymbol{\eta}_{ji}^-, \quad \boldsymbol{\theta}_{ij} = -\boldsymbol{\theta}_{ji}, \\
&r_{ij} = A(L_{ij})/(2A(C_i)), \quad \textstyle\sum_{j \in N_i} r_{ij} = 1 \quad \text{and} \quad \textstyle\sum_{j \in N_i} \boldsymbol{\theta}_{ij} = 0.
\end{aligned}
$$

For our convergence proof we need a nondegeneracy-condition on the triangulation.

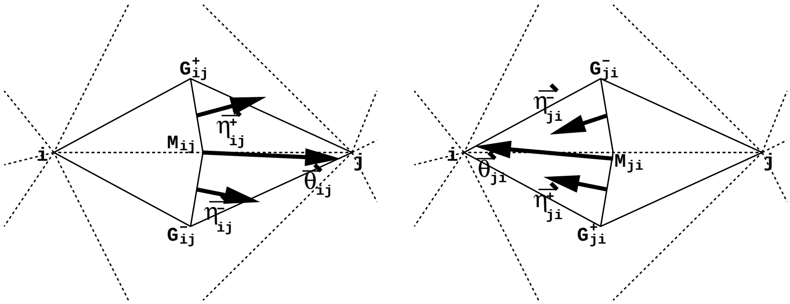


Fig. 2. Normals to C_i and L_{ij}

Definition 3.4. *A conform unstructured triangulation $\mathcal{T}$ will be called an (a,b)-**nondegenerate triangulation**, if the length of its edges are bounded, i.e. the supremum h of these exists, and the areas of all triangles $D \in \mathcal{T}$ are bounded by*

$$ah^2 < A(D) < bh^2.$$

From these bounds we gain some geometrical estimates.

Lemma 3.5. *Let $\mathcal{T}$ be an (a,b)-nondegenerate triangulation. Then for all $i \in I, j \in N_i$ the following geometrical estimates hold, where $diam$ denotes the diameter of a polygon:*

$$\text{(7)} \qquad \begin{aligned} &a)\ \frac{2}{3}ah^2 \leq A\left(L_{ij}\right) \leq \frac{2}{3}bh^2, \\ &b)\ \left|\boldsymbol{\eta}_{ij}^-\right| \leq \frac{1}{3}h, \quad \left|\boldsymbol{\eta}_{ij}^+\right| \leq \frac{1}{3}h, \quad \left|\boldsymbol{\theta}_{ij}\right| \leq \frac{2}{3}h\ , \\ &c)\ diam\left(C_i\right) \leq \frac{4}{3}h \quad \textit{and} \quad A\left(C_i\right) \leq \frac{4}{9}\pi h^2. \end{aligned}$$

The proofs are trivial remembering that G_{ij}^+ resp. G_{ij}^- are centers of gravity.

Finally we define the discrete solution obtained by the staggered Lax-Friedrichs scheme. In [3,4] these formulas are obtained by applying evolution-projection ideas on the finite volume cells C_i and L_{ij}.

Definition 3.6. (StgLxF-scheme) *Let $\mathcal{T}$ be an (a,b)-nondegenerate triangulation, $\Delta t > 0$. We define for all $i \in I, j \in N_i, n \in 2\mathbb{N}$*

$$\text{(8)} \qquad u_i^0 := \frac{1}{A\left(C_i\right)} \int_{C_i} u_0,$$

$$\text{(9)} \qquad u_{ij}^{n+1} := \frac{1}{2}\left(u_i^n + u_j^n\right) - \frac{\Delta t}{A\left(L_{ij}\right)}\left(\mathbf{f}\left(u_j^n\right) - \mathbf{f}\left(u_i^n\right)\right)\boldsymbol{\theta}_{ij},$$

$$\text{(10)} \qquad u_i^{n+2} := \sum_{j \in N_i} r_{ij} u_{ij}^{n+1} - \frac{\Delta t}{A\left(C_i\right)} \sum_{j \in N_i} \mathbf{f}\left(u_{ij}^{n+1}\right)\boldsymbol{\theta}_{ij}.$$

These values define the numerical solution

$$u_h(x,t) := \begin{cases} u_i^n & for \quad (x,t) \in C_i^o \times [n\Delta t, (n+1)\Delta t) \\ u_{ij}^{n+1} & for \quad (x,t) \in L_{ij}^o \times [(n+1)\Delta t, (n+2)\Delta t) \end{cases} . \tag{11}$$

4. Convergence

We formulate the main result of this paper.

Theorem 4.1. (Convergence of StgLxF-scheme) *Let* $f_1, f_2 \in C^1(\mathbb{R}), u_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *with* $B := \|u_0\|_{L^\infty}$ *and* $T > 0$. *We define*

$$L := \max_{u \in [-B,B], i=1,2} \left| f_i'(u) \right| .$$

Let further $(\mathcal{T}_k)_{k \in \mathbb{N}}$ *be a sequence of uniformly* (a,b)*-nondegenerate triangulations in the sense of Def.3.4 (*a*,*b *independent of* k*) with*

$$\lim_{k \to \infty} h_k = 0.$$

Let β, γ *be constants with the relation* $0 < \gamma < \beta < \frac{a}{4}$. *Let* $(\Delta t_k)_{k \in \mathbb{N}}$ *be a sequence of timesteps, such that for all* $k \in \mathbb{N}$ *the CFL-condition*

$$\frac{\Delta t_k}{h_k} L \leq \beta \tag{12}$$

and the lower bound

$$\gamma \leq \frac{\Delta t_k}{h_k} L \tag{13}$$

hold. $(u_{h_k})_{k \in \mathbb{N}}$ *denotes the associated sequence of numerical solutions defined by the StgLxF-scheme (8)–(11).*

Then the sequence $(u_{h_k})_{k \in \mathbb{N}}$ *converges in* $L^q_{loc}\left(\mathbb{R}^2 \times [0,T]\right)$ *for all* $1 \leq q < \infty$ *strongly towards the unique weak entropy solution of the Cauchy-problem (1) and (2).*

Remark 4.2. Although Theorem 4.1 is restricted to pure convergence we stress that an a–priori error estimate of the type $\|u - u_h\|_{L^1} \leq Ch^{1/4}$ can be achieved easily following the lines of Bouchut-Perthame [6], for instance. Furthermore, generalizations of 4.1 to higher-order schemes as in [12] or to the case of weakly coupled systems[17] can be realized. We believe that the proof can also be extended to the three-dimensional scheme on unstructured grids recently presented by Arminjon [1].

This formulation of convergence allows the existence of $\gamma > 0$ just in the case of $L > 0$, which means $\mathbf{f}' \not\equiv (0,0)^{\mathrm{T}}$. For the trivial case $L = 0$ the proof is identical after choosing arbitrary $L > 0$. The proof of Thm. 4.1 consists of showing that the conditions 2.6 a) to e) are satisfied by the sequence $(u_{h_k})_{k\in\mathbb{N}}$. This will be done in the sequel. In particular a) will be proven by Prop. 4.3, b) is obtained by Prop. 4.6, c) and d) are the statement of Prop. 4.10 and e) is a consequence of Prop. 4.7.

Proposition 4.3. (L^∞-bound) *Let the assumptions of Thm.4.1 be valid,* u_h *be an element of the sequence of numerical solutions. Then we have for all* $n \in \mathbb{N}, i \in I, j \in N_i$

$$|u_i^0| \le B, \tag{14}$$

$$\min\left\{u_i^n, u_j^n\right\} \le u_{ij}^{n+1} \le \max\left\{u_i^n, u_j^n\right\}, \tag{15}$$

$$\min_{j\in N_i}\left\{u_{ij}^{n+1}\right\} \le u_i^{n+2} \le \max_{j\in N_i}\left\{u_{ij}^{n+1}\right\}, \tag{16}$$

and therefore condition 2.6 a) is satisfied by $C_1 := B$.

Proof. (14) follows directly from the definition of u_i^0 as average of the function u_0, which is bounded in L^∞ by B.

(16) needs some calculations: First we obtain with (10) for an arbitrary $v \in [-B, B]$

$$v - u_i^{n+2} = v - \sum_{j\in N_i} r_{ij} u_{ij}^{n+1} - \frac{\Delta t}{A(C_i)} \sum_{j\in N_i} -\mathbf{f}\left(u_{ij}^{n+1}\right)\boldsymbol{\theta}_{ij}.$$

As $\sum_{j\in N_i} r_{ij} = 1$ and $\sum_{j\in N_i} \boldsymbol{\theta}_{ij} = 0$ we can insert terms dependent of v in both sums. Therefore the right hand is equivalent to

$$\begin{aligned}
&\sum_{j\in N_i} r_{ij}\left(v - u_{ij}^{n+1}\right) - \frac{\Delta t}{A(C_i)} \sum_{j\in N_i}\left(\mathbf{f}(v) - \mathbf{f}\left(u_{ij}^{n+1}\right)\right)\boldsymbol{\theta}_{ij} \\
&= \sum_{j\in N_i} r_{ij}\left(v - u_{ij}^{n+1}\right) - \frac{\Delta t}{A(C_i)} \sum_{j\in N_i}\begin{pmatrix} f_1(v) - f_1(u_{ij}^{n+1}) \\ f_2(v) - f_2(u_{ij}^{n+1})\end{pmatrix}\boldsymbol{\theta}_{ij}.
\end{aligned}$$

For some ξ_{j1}, ξ_{j2} between v, u_{ij}^{n+1} we continue with

$$= \sum_{j\in N_i} r_{ij}\left(v - u_{ij}^{n+1}\right) - \frac{\Delta t}{A(C_i)} \sum_{j\in N_i}\left(v - u_{ij}^{n+1}\right)\begin{pmatrix} f_1'(\xi_{j1}) \\ f_2'(\xi_{j2})\end{pmatrix}\boldsymbol{\theta}_{ij}.$$

We conclude with an equivalence for any $v \in [-B, B]$ with appropriate $\xi_{j1}, \xi_{j2} \in [-B, B]$:

$$v - u_i^{n+2} = \sum_{j\in N_i}\left(v - u_{ij}^{n+1}\right)\left[r_{ij} - \frac{\Delta t}{A(C_i)}\begin{pmatrix} f_1'(\xi_{j1}) \\ f_2'(\xi_{j2})\end{pmatrix}\boldsymbol{\theta}_{ij}\right]. \tag{17}$$

For (16) it is sufficient to prove

$$\max_{j\in N_i}\left\{u_{ij}^{n+1}\right\} - u_i^{n+2} \geq 0 \qquad \text{and} \tag{18}$$

$$\min_{j\in N_i}\left\{u_{ij}^{n+1}\right\} - u_i^{n+2} \leq 0. \tag{19}$$

For (18) it is with $v := \max_{j\in N_i}\{u_{ij}^{n+1}\}$ in (17) sufficient to prove, that all addends on the right are positive. We already have for all $j \in N_i$: $v - u_{ij}^{n+1} \geq 0$, therefore the first factor in the sum is positive. It remains to show that the factor in angular brackets is also positive.

For (19) we have with $v := \min_{j\in N_i}\{u_{ij}^{n+1}\}$ that for all $j \in N_i$ holds: $v - u_{ij}^{n+1} \leq 0$, such that the first factor in the sum of (17) is negative. Sufficient for the inequality in question is again that the factor in angular brackets is positive.

This follows from geometrical estimates, the Lipschitz-bound of $\mathbf{f}$ and the CFL-condition (12), as for any $\xi_1, \xi_2 \in [-B, B]$ holds

$$\begin{aligned}\frac{A(L_{ij})}{A(C_i)} \cdot \frac{\Delta t}{A(L_{ij})}\left|\begin{pmatrix} f_1'(\xi_1) \\ f_2'(\xi_2)\end{pmatrix}\right| |\boldsymbol{\theta}_{ij}| &\leq 2r_{ij}\cdot\frac{\Delta t}{\frac{2}{3}ah^2}\cdot 2L\cdot\frac{2}{3}h \\ &\leq 2r_{ij}\cdot\frac{2}{a}\beta < r_{ij}.\end{aligned} \tag{20}$$

Equation (15) can be proven analogously.

To verify conditions 2.6 b) to e) we first need two discrete entropy inequalities.

Proposition 4.4. (Discrete entropy inequalities) *Let the assumptions of Thm. 4.1 be valid, u_h be an element of the sequence of numerical solutions. Then for all $n \in \mathbb{N}, i \in I, j \in N_i$ and entropy pairs $(U, \mathbf{F})$ hold*

$$\begin{aligned}U\left(u_{ij}^{n+1}\right) \leq{}& \frac{1}{2}\left(U\left(u_i^n\right) + U\left(u_j^n\right)\right) \\ &- \frac{\Delta t}{A\left(L_{ij}\right)}\left(\mathbf{F}\left(u_j^n\right) - \mathbf{F}\left(u_i^n\right)\right)\boldsymbol{\theta}_{ij},\end{aligned} \tag{21}$$

$$\begin{aligned}U\left(u_i^{n+2}\right) \leq{}& \sum_{j\in N_i} r_{ij} U\left(u_{ij}^{n+1}\right) \\ &- \frac{\Delta t}{A\left(C_i\right)}\sum_{j\in N_i}\mathbf{F}\left(u_{ij}^{n+1}\right)\boldsymbol{\theta}_{ij}.\end{aligned} \tag{22}$$

This Proposition turns out to be the most important point in the proof, as it is fundamental for all further results. The problem of the initially unknown structure of the entropy inequalities turns out to be perfectly solved by the inequalities (21),(22): first these inequalities are actually satisfied by the scheme and secondly they allow to derive the further properties which are sufficient for convergence of the scheme.

Proof. We just prove the second inequality, as the proof for the first one is similar. Let $(U, \mathbf{F})$ be an entropy pair, $n \in 2\mathbb{N}, i \in I$ fixed. We sort and enumerate the set $\{u_{ij}^{n+1}\}_{j \in N_i}$ by size.

For this we take $m := |N_i|$. Let $\alpha : \{1, \ldots, m\} \to N_i$ be a bijection, such that with $u_l := u_{i\alpha(l)}^{n+1}$ holds

$$u_1 \leq u_2 \leq \ldots \leq u_m. \tag{23}$$

This map α also induces an enumeration on r_{ij} and $\boldsymbol{\theta}_{ij}$. For the second step of the scheme (10) we obtain the alternative representation

$$u_i^{n+2} = \sum_{l=1}^{m} \left(r_l u_l - \frac{\Delta t}{A(C_i)} \mathbf{f}(u_l) \boldsymbol{\theta}_l \right). \tag{24}$$

The inequality in question is written accordingly

$$U\left(u_i^{n+2}\right) \leq \sum_{l=1}^{m} \left(r_l U(u_l) - \frac{\Delta t}{A(C_i)} \mathbf{F}(u_l) \boldsymbol{\theta}_l \right). \tag{25}$$

To reformulate this once more, we define

$$q(t_1, \ldots, t_m) := \sum_{l=1}^{m} \left(r_l t_l - \frac{\Delta t}{A(C_i)} \mathbf{f}(t_l) \boldsymbol{\theta}_l \right) \quad \text{and}$$

$$p(t_1, \ldots, t_m) := U(q(t_1, \ldots, t_m)) - \sum_{l=1}^{m} \left(r_l U(t_l) - \frac{\Delta t}{A(C_i)} \mathbf{F}(t_l) \boldsymbol{\theta}_l \right).$$

This implies $q(u_1, \ldots, u_m) = u_i^{n+2}$, compare (24). Thus (25) is equivalent to the statement

$$p(u_1, \ldots, u_m) \leq 0. \tag{26}$$

This will be proven in the following.

As p is differentiable it is sufficient for (26) to find points $P_1, \ldots, P_m \in \mathbb{R}^m$ such that

i) $p(P_1) = 0$, ii) $P_m = (u_1, \ldots, u_m)^{\mathrm{T}}$,
iii) for all $k = 2, \ldots, m$ and all $P \in \overline{P_{k-1} P_k}$ holds

$$\nabla p|_P \cdot (P_k - P_{k-1}) \leq 0.$$

These conditions are satisfied by the choice

$$\begin{aligned} P_1 &:= (u_1, \ldots, u_1)^{\mathrm{T}} \\ P_2 &:= (u_1, u_2, \ldots, u_2)^{\mathrm{T}} \\ &\vdots \quad \vdots \\ P_{m-1} &:= (u_1, \ldots, u_{m-1}, u_{m-1})^{\mathrm{T}} \\ P_m &:= (u_1, \ldots, u_m)^{\mathrm{T}}. \end{aligned}$$

It remains to check the conditions i) and iii) because ii) holds trivially. Proof of i): We obtain with Remark 3.3

$$\begin{aligned} q(u_1, \ldots, u_1) &= \sum_{l=1}^{m} \left(r_l u_1 - \frac{\Delta t}{A(C_i)} \mathbf{f}\ (u_1)\, \boldsymbol{\theta}_l \right) \\ &= u_1 \sum_{l=1}^{m} r_l - \frac{\Delta t}{A(C_i)} \mathbf{f}(u_1) \sum_{l=1}^{m} \boldsymbol{\theta}_l \quad = \quad u_1. \end{aligned}$$

This is used in the definition of p and we obtain with Remark 3.3

$$\begin{aligned} p(P_1) &= p(u_1, \ldots, u_1) \\ &= U(q(u_1, \ldots, u_1)) - \sum_{l=1}^{m} \left(r_l U(u_1) - \frac{\Delta t}{A(C_i)} \mathbf{F}(u_1)\, \boldsymbol{\theta}_l \right) \\ &= U(u_1) - U(u_1) \quad = \quad 0. \end{aligned}$$

It remains to prove iii). Choose $k \in \{2, \ldots, m\}, \quad P \in \overline{P_{k-1} P_k}$, that means

$$P = (u_1, \ldots, u_{k-1}, u, \ldots, u)^{\mathrm{T}} \quad \text{for some} \quad u \in [u_{k-1}, u_k]. \tag{27}$$

We reformulate the inequality of iii) using the notation $\mathbf{e}_l$ for the l-th unit-vector of $\mathbb{R}^m$. This yields for the difference-vector

$$P_k - P_{k-1} = (u_k - u_{k-1}) \sum_{l=k}^{m} \mathbf{e}_l,$$

and herewith

$$\begin{aligned} \nabla p|_P \cdot (P_k - P_{k-1}) &= (u_k - u_{k-1}) \sum_{l=k}^{m} \nabla p|_P \cdot \mathbf{e}_l \\ &= (u_k - u_{k-1}) \sum_{l=k}^{m} \left. \frac{\partial}{\partial t_l} p \right|_P . \end{aligned}$$

As the first factor is nonnegative due to (23), it is sufficient to show that

$$\text{for all} \quad l = k, \ldots, m \quad \text{holds} \quad \frac{\partial}{\partial t_l} p \bigg|_P \leq 0. \tag{28}$$

Let $l \in \{k, \ldots, m\}$ be fixed. We obtain by definition of p

$$\frac{\partial}{\partial t_l} p(t_1, \ldots, t_m) = U'(q(t_1, \ldots, t_m)) \frac{\partial}{\partial t_l} q(t_1, \ldots, t_m) - \left(r_l U'(t_l) - \frac{\Delta t}{A(C_i)} \mathbf{F}'(t_l) \boldsymbol{\theta}_l \right).$$

From the definition of q we obtain

$$\frac{\partial}{\partial t_l} q(t_1, \ldots, t_m) = r_l - \frac{\Delta t}{A(C_i)} \mathbf{f}'(t_l) \boldsymbol{\theta}_l.$$

This yields together with $\mathbf{F}'(t) = U'(t)\mathbf{f}'(t)$ in the partial derivative of p

$$\begin{aligned} &\frac{\partial}{\partial t_l} p(t_1, \ldots, t_m) \\ &= U'(q(t_1, \ldots, t_m)) \left(r_l - \frac{\Delta t}{A(C_i)} \mathbf{f}'(t_l) \boldsymbol{\theta}_l \right) - \\ &\quad \left(r_l U'(t_l) - \frac{\Delta t}{A(C_i)} U'(t_l) \mathbf{f}'(t_l) \boldsymbol{\theta}_l \right) \\ &= \Big(U'(q(t_1, \ldots, t_m)) - U'(t_l) \Big) \left(r_l - \frac{\Delta t}{A(C_i)} \mathbf{f}'(t_l) \boldsymbol{\theta}_l \right). \end{aligned}$$

There exists some ξ between $q(t_1, \ldots, t_m)$ and t_l, such that

$$\frac{\partial}{\partial t_l} p(t_1, \ldots, t_m) = U''(\xi) (q(t_1, \ldots, t_m) - t_l) \left(r_l - \frac{\Delta t}{A(C_i)} \mathbf{f}'(t_l) \boldsymbol{\theta}_l \right). \tag{29}$$

The first factor is positive with convexity of U, the last one is positive with the calculations in (20). Therefore it is sufficient to prove that the middle factor is negative if evaluated in P. With definition of q we rewrite it again

$$\begin{aligned} q(t_1, \ldots, t_m) - t_l &= \sum_{j=1}^{m} \left(r_j t_j - \frac{\Delta t}{A(C_i)} \mathbf{f}(t_j) \boldsymbol{\theta}_j \right) - t_l \\ &= \sum_{j=1}^{m} \left(r_j (t_j - t_l) - \frac{\Delta t}{A(C_i)} (\mathbf{f}(t_j) - \mathbf{f}(t_l)) \boldsymbol{\theta}_j \right) \\ &= \sum_{j=1}^{m} (t_j - t_l) \left(r_j - \frac{\Delta t}{A(C_i)} \begin{pmatrix} f_1'(\xi_{j1}) \\ f_2'(\xi_{j2}) \end{pmatrix} \boldsymbol{\theta}_j \right) \end{aligned} \tag{30}$$

for certain ξ_{j1}, ξ_{j2} between t_j and t_l. Evaluation in P yields $t_j|_P = u_j$ or u and $t_l|_P = u$ with (27). Therefore we obtain by (20) again that the last factors in (30) are positive if evaluated in P. The first factors seperate in two cases:
If $j \le k-1$ we have $t_j - t_l|_P = u_j - u \le u_j - u_{k-1} \le 0$ due to (27) and (23).
If $j \ge k$ we have $t_j - t_l|_P = u - u = 0$. Thus (28) holds and herewith the proposition.

Notation 4.5. *Let $D \subset \mathbb{R}^2$ be a disc. We denote $I_D := I \cap D$ as the set of vertices in D. Let a, b, h_0 be nonnegative constants. Then there exists a constant C_D, such that for all (a,b)-nondegenerate triangulations with $h \le h_0$ holds (cf. Not. 3.1)*

$$\sum_{i \in I_D} |N_i| \le C_D \frac{1}{h^2}. \tag{31}$$

Regarding the geometrical estimates of Lemma 3.5 the proof is obvious. From the entropy inequalities now follows the uniform L^1-stability.

Proposition 4.6. (Uniform L^1-stability) *Let the assumptions of Thm. 4.1 be valid, u_h be an element of the sequence of numerical solutions. Then for all $n \in \mathbb{N}, i \in I, j \in N_i$ hold*

$$\sum_{i \in I} A(C_i) \left|u_i^0\right| \le \|u_0\|_{L^1(\mathbb{R}^2)}, \tag{32}$$

$$\sum_{i \in I, j \in N_i} \frac{1}{2} A(L_{ij}) \left|u_{ij}^{n+1}\right| \le \sum_{i \in I} A(C_i) \left|u_i^n\right|, \tag{33}$$

$$\sum_{i \in I} A(C_i) \left|u_i^{n+2}\right| \le \sum_{i \in I, j \in N_i} \frac{1}{2} A(L_{ij}) \left|u_{ij}^{n+1}\right|. \tag{34}$$

and therefore condition 2.6 b) is satisfied by $C_2 := \|u_0\|_{L^1(\mathbb{R}^2)}$.

Proof. (32) again follows directly from the definition of the initial-values (8).

Proof of (33): Let $(U_m)_{m \ge 1}$ be a sequence of nonnegative convex functions with Lipschitz-bound 1 and which approximate $|\cdot|$ from below with $0 \le |u| - U_m(u) \le 1/m$ for all $u \in \mathbb{R}$. For such entropy U_m and entropy flux $\mathbf{F}_m$, we multiply the first discrete entropy inequality (21) with $A(L_{ij})/2$ and sum it over $i \in I, j \in N_i$. This yields

$$\sum_{i \in I, j \in N_i} \frac{1}{2} A(L_{ij}) U_m\left(u_{ij}^{n+1}\right)$$

$$\leq \sum_{i\in I, j\in N_i} \frac{1}{2} A(L_{ij}) \left(\frac{1}{2} U_m(u_i^n) + \frac{1}{2} U_m(u_j^n)\right)$$
$$- \sum_{i\in I, j\in N_i} \frac{1}{2} \Delta t \left(\mathbf{F}_m(u_j^n) - \mathbf{F}_m(u_i^n)\right) \boldsymbol{\theta}_{ij}$$
$$=: T_1 - T_2. \tag{35}$$

We restrict the sum on the left hand to $i \in I_D$ for an arbitrary disc D with radius R and use the properties of U_m:

$$\sum_{i\in I, j\in N_i} \frac{1}{2} A(L_{ij}) U_m\left(u_{ij}^{n+1}\right) \geq \sum_{i\in I_D, j\in N_i} \frac{1}{2} A(L_{ij}) U_m\left(u_{ij}^{n+1}\right)$$
$$\geq \sum_{i\in I_D, j\in N_i} \frac{1}{2} A(L_{ij}) \left|u_{ij}^{n+1}\right|$$
$$- \sum_{i\in I_D, j\in N_i} \frac{1}{2} A(L_{ij}) \frac{1}{m} =: T_3 - T_4.$$

We obtain with (35)

$$T_3 - T_4 \leq T_1 - T_2.$$

The terms on the right hand are absolutely convergent, which can proven easily with the help of $\sum_{i\in I} A(C_i) |u_i^n| < \infty$. Rewriting the sums in T_1 and T_2 by swapping i, j directly yields

$$T_1 \leq \sum_{i\in I} A(C_i) |u_i^n|, \quad T_2 = 0.$$

In the limit $m \to \infty$, T_4 disappears, thus T_3 is bounded independent of D by T_1. By the limit $R \to \infty$ we obtain (33). (34) can be proven analogously.

Proposition 4.7. (L^2-stability) *Let the assumptions of Thm.4.1 be valid, u_h be an element of the sequence of numerical solutions, $U(u) = u^2/2$. Then*

$$\|U(u_h(\cdot,0))\|_{L^1(\mathbb{R}^2)} \leq \|U(u_0))\|_{L^1(\mathbb{R}^2)} \leq \infty$$

and for all $0 \leq t_1 \leq t_2 \leq T$

$$\|U(u_h(\cdot,t_2))\|_{L^1(\mathbb{R}^2)} \leq \|U(u_h(\cdot,t_1))\|_{L^1(\mathbb{R}^2)}$$

hold. Therefore condition 2.6 e) is valid.

Proof. The first statement follows from the definition of the initial values and the relation $\|u_0\|^2_{L^2(\mathbb{R}^2)} \leq \|u_0\|_{L^1(\mathbb{R}^2)} \|u_0\|_{L^\infty}$. The second statement follows identical to the proof of the previous proposition: We obtain (35) containing $U(u) = u^2/2$ instead of U_m. Again absolute convergence of T_1 and T_2 can be proven and rearranging the sums yields the result for the first step of the scheme. The second step follows similarly.

For the proof of the remaining conditions 2.6 c) and d) we need a weak BV-estimate. This is based on a more accurate estimate of the entropy dissipation for a quadratic entropy.

Lemma 4.8. (Entropy dissipation) *Let the assumptions of Thm.4.1 be valid. Then there exists a nonnegative constant* C *such that for all numerical solutions* u_h *of the sequence, all* $n \in \mathbb{N}, i \in I, j \in N_i$ *and entropy pairs* $(U, \mathbf{F})$ *with* $U(u) = u^2/2$ *holds*

$$U\left(u_{ij}^{n+1}\right) - \frac{1}{2}\left(U\left(u_i^n\right) + U\left(u_j^n\right)\right) + \frac{\Delta t}{A\left(L_{ij}\right)}\left(\mathbf{F}\left(u_j^n\right) - \mathbf{F}\left(u_i^n\right)\right)\boldsymbol{\theta}_{ij} \leq -C\left(u_i^n - u_j^n\right)^2. \tag{36}$$

The constant might be chosen as $C = C(a, \beta) = \frac{1}{2}\left(\frac{1}{2} - \frac{2}{a}\beta\right)^2$.

Proof. We define similar as in the proof of Prop. 4.4

$$\begin{aligned} q(t_1, t_2) &:= \frac{1}{2}(t_1 + t_2) - \frac{\Delta t}{A\left(L_{ij}\right)}\left(\mathbf{f}(t_2) - \mathbf{f}(t_1)\right)\boldsymbol{\theta}_{ij}. \\ p(t_1, t_2) &:= U\left(q(t_1, t_2)\right) - \frac{1}{2}\left(U(t_1) + U(t_2)\right) \\ &\quad + \frac{\Delta t}{A\left(L_{ij}\right)}\left(\mathbf{F}(t_2) - \mathbf{F}(t_1)\right)\boldsymbol{\theta}_{ij}. \end{aligned}$$

Further calculation yields a representation similar to (29), (30)

$$\begin{aligned} \frac{\partial}{\partial t_2} p(t_1, t_2) = &\left(\frac{1}{2} - \frac{\Delta t}{A\left(L_{ij}\right)}\mathbf{f}'(t_2)\boldsymbol{\theta}_{ij}\right) U''(\xi)\left(t_1 - t_2\right) \\ &\cdot\left(\frac{1}{2} + \frac{\Delta t}{A\left(L_{ij}\right)}\begin{pmatrix} f_1'(\xi_1) \\ f_2'(\xi_2) \end{pmatrix}\boldsymbol{\theta}_{ij}\right), \end{aligned} \tag{37}$$

for certain ξ, ξ_1, ξ_2 between t_1 and t_2. With $r(t_1, t_2) := -C(t_1 - t_2)^2$ it is sufficient to show that $p(t_1, t_2) \leq r(t_1, t_2)$. As p and r are identical zero for $t_2 = t_1$, this is equivalent to proving

$$(t_1 - t_2)\frac{\partial}{\partial t_2} p(t_1, t_2) \geq (t_1 - t_2)\frac{\partial}{\partial t_2} r(t_1, t_2) \tag{38}$$

for all $t_1, t_2 \in [-B, B]$. We rewrite the left hand with (37). Using (20) we see that the first and last factor in (37) are greater or equal to $\frac{1}{2} - \frac{2}{a}\beta$. Therefore we obtain with $U'' \equiv 1$ and C

$$(t_1 - t_2)\frac{\partial}{\partial t_2} p(t_1, t_2) \geq 2C(t_1 - t_2)^2.$$

This is exactly the statement of (38).

Using this estimate, we now derive a weak BV-estimate. This kind of estimate is weaker than a BV-estimate, but strong enough to obtain convergence. A similar estimate was derived in the proof for the case of the linear equation [5], called an estimate on the weighted total variation.

Proposition 4.9. (Weak BV-estimate) *Let the assumptions of Thm. 4.1 be valid, $D \subset \mathbb{R}^2$ be a disc. Then there exists a constant C, such that for all numerical solutions u_h of the sequence and corresponding $h, \Delta t$ with $N := max\{n \in 2\mathbb{N} | n\Delta t \leq T\}$ holds*

$$\sum_{\substack{i \in I_D, j \in N_i \\ n \in 2\mathbb{N}, n \leq N}} h^2 \left| u_i^n - u_j^n \right| \leq C h^{-\frac{1}{2}}. \tag{39}$$

Proof. For every $n \in 2\mathbb{N}$ and C as given in Lemma 4.8

$$C \sum_{i \in I_D, j \in N_i} A\left(L_{ij}\right) \left(u_i^n - u_j^n\right)^2$$

$$\leq \sum_{i \in I, j \in N_i} \frac{1}{2} A\left(L_{ij}\right) \left(\left(u_i^n\right)^2 - \left(u_{ij}^{n+1}\right)^2 \right), \tag{40}$$

$$0 \leq \sum_{i \in I, j \in N_i} \frac{1}{2} A\left(L_{ij}\right) \left(\left(u_{ij}^{n+1}\right)^2 - \left(u_i^{n+2}\right)^2 \right) \tag{41}$$

hold. (40) follows by multiplying (36) with $A(L_{ij})$ and summing over $i \in I, j \in N_i$. The sum over $(u_j^n)^2$ is rewritten as sum over $(u_i^n)^2$ by swapping i, j, the sum over the entropy fluxes equals zero and the sum of the squares of the differences is restricted to $i \in I_D, j \in N_i$.

Equation (41) follows by multiplying the second entropy inequality (22) by $2A(C_i)$, using $U(u) = u^2/2, r_{ij} A(C_i) = A(L_{ij})/2$ and $\sum_{j \in N_i} r_{ij} = 1$. This is summed up over all $i \in I$, the sum over the fluxes again cancels after splitting and swapping i, j, as $\boldsymbol{\theta}_{ij} = -\boldsymbol{\theta}_{ji}$:

$$\sum_{i \in I, j \in N_i} 2\Delta t \mathbf{F}\left(u_{ij}^{n+1}\right) \boldsymbol{\theta}_{ij} = \sum_{i \in I, j \in N_i} \Delta t \mathbf{F}\left(u_{ij}^{n+1}\right) \boldsymbol{\theta}_{ij} + \sum_{i \in I, j \in N_i} \Delta t \mathbf{F}\left(u_{ij}^{n+1}\right) \boldsymbol{\theta}_{ji}.$$

Let $\sum_{i,j,n}$ denote the sum over all $i \in I_D, j \in N_i, n \in 2\mathbb{N}, n \leq N$ and $\sum_n$ denote the sum over these n. We sum the inequalities (40) and (41) over all $n \in 2\mathbb{N}, n \leq N$, by this the right hand simplifies as the addends cancel pairwise except two remaining terms

$$C\sum_{i,j,n} A\left(L_{ij}\right)\left(u_i^n - u_j^n\right)^2 \leq \sum_{i\in I, j\in N_i} \frac{1}{2} A\left(L_{ij}\right)\left(\left(u_i^0\right)^2 - \left(u_i^{N+2}\right)^2\right) \tag{42}$$

Let us denote the term on the right hand by RH, the left hand by LH. Proposition 4.3 and 4.6 yield an upper bound for RH:

$$RH \leq \sum_{i\in I, j\in N_i} \frac{1}{2} A\left(L_{ij}\right)\left(u_i^0\right)^2 \leq B\left\|u_h(\cdot,0)\right\|_{L^1(\mathbb{R}^2)} \leq B\left\|u_0\right\|_{L^1(\mathbb{R}^2)}.$$

Further the left hand can be estimated from below using (7):

$$C\frac{2}{3}a\sum_{i,j,n} h^2\left(u_i^n - u_j^n\right)^2 \leq LH.$$

We conclude with a new constant C

$$\sum_{i,j,n} h^2\left(u_i^n - u_j^n\right)^2 \leq C. \tag{43}$$

The statement of the proposition follows easily by using the Cauchy-Schwarz inequality:

$$\sum_{i,j,n} h^2\left|u_i^n - u_j^n\right| \leq \sqrt{\sum_{i,j,n} h^2}\sqrt{\sum_{i,j,n} h^2\left(u_i^n - u_j^n\right)^2}.$$

The last factor can be bounded with help of (43) by $\sqrt{C}$, the first one can be estimated with assumption $N \leq T/\Delta t$, the inverse CFL-property (13) $L\Delta t/h \geq \gamma$, (31) and $h_0 := \max_{k\in\mathbb{N}} h_k$

$$\begin{aligned}\sum_n \sum_{i\in I_D, j\in N_i} h^2 \leq \sum_n C_D \frac{1}{h^2} h^2 &= \left(\frac{N}{2}+1\right) C_D \\ \leq \left(\frac{TL}{2\gamma h} + \frac{h_0}{h}\right) C_D &= \left(\frac{TL}{2\gamma} + h_0\right) C_D \frac{1}{h} =: C'\frac{1}{h}.\end{aligned}$$

This yields the statement of the proposition

$$\sum_{i,j,n} h^2\left|u_i^n - u_j^n\right| \leq \sqrt{C'C}h^{-\frac{1}{2}},$$

as both C and C' are independent of the choice of u_h.

With this weak BV-estimate we can prove a last proposition, which completes the convergence proof.

Proposition 4.10. (Weak consistency) *Let the assumptions of Thm. 4.1 be valid,* $\varphi \in C_0^\infty\left(\mathbb{R}^2 \times [0,T)\right)$*. Then nonnegative constants* C_φ *and* $C_{U,\mathbf{F},\varphi}$ *exist, such that for all* $k \in \mathbb{N}$ *the numerical solution* u_{h_k} *satisfies*

$$\left| \int_0^{\mathrm{T}} \int_{\mathbb{R}^2} \left(u_{h_k} \varphi_t + \mathbf{f}\left(u_{h_k}\right) \nabla \varphi \right) + \int_{\mathbb{R}^2} u_0 \varphi(\cdot, 0) \right| \le C_\varphi h_k^{\frac{1}{2}},$$

and for all entropy pairs $(U, \mathbf{F})$*,* φ *with* $\varphi(x,0) \equiv 0$*,* $\varphi \ge 0$ *holds*

$$\int_0^{\mathrm{T}} \int_{\mathbb{R}^2} \left(U\left(u_{h_k}\right) \varphi_t + \mathbf{F}\left(u_{h_k}\right) \nabla \varphi \right) \ge -C_{U,\mathbf{F},\varphi} h_k^{\frac{1}{2}}.$$

Thus conditions 2.6 c) and d) are satisfied by $\kappa := 1/2$*.*

Proof. Let u_h be an element of the sequence of numerical solutions, $(U, \mathbf{F})$ be an arbitrary entropy pair. We first give some notations which will be used in the proof.

- Let N be defined as in Prop. 4.9, $N := max\left\{ n \in 2\mathbb{N} \,|\, n\Delta t \le T \right\}$.
- Let $h_0 := \max_{k \in \mathbb{N}} h_k$.
- Let $D \subset \mathbb{R}^2$ be a disc, which covers the support of $\varphi(\cdot, t)$ for all t and is large enough, that φ vanishes on all boundary-cells C_i of the set $\{C_i\}_{i \in I_D}$ for all $t \in [0,T]$ and all triangulations $\mathcal{T}$ from the sequence $(\mathcal{T}_k)_{k \in \mathbb{N}}$.
- Let C_φ be an L^∞-bound of φ and all of its derivatives up to degree 2.
- We denote $\varphi^n := \varphi(\cdot, n\Delta t)$, $\quad \varphi_t^n := \varphi_t(\cdot, n\Delta t)$.
- S_i denotes the center of gravity of cell C_i for all $i \in I$.
- Let $\varphi_i^n := \varphi^n\left(S_i\right)$ for all $n \in 2\mathbb{N}, i \in I$.
- The frequently used sums are abbreviated by $\sum_n := \sum_{n \in 2\mathbb{N}, n \le N}$, $\sum_i := \sum_{i \in I_D}$, $\sum_{i,n} := \sum_i \sum_n$, $\sum_{i,j} := \sum_i \sum_{j \in N_i}$ and $\sum_{i,j,n} := \sum_{i,j} \sum_n$.

With these notations the statement of the proposition follows from

a) $$-\sum_{i,n} 2\Delta t A\left(C_i\right) \frac{u_i^{n+2} - u_i^n}{2\Delta t} \varphi_i^n = \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} u_h \varphi_t + \int_{\mathbb{R}^2} u_0 \varphi^0 + \mathcal{O}\left(h^{\frac{1}{2}}\right),$$

b) $$-\sum_{i,n} A\left(C_i\right) \left(u_i^{n+2} - u_i^n\right) \varphi_i^n = -\int_{\mathbb{R}^+} \int_{\mathbb{R}^2} \mathbf{f}\left(u_h\right) \nabla \varphi + \mathcal{O}\left(h^{\frac{1}{2}}\right),$$

c) $$-\sum_{i,n} 2\Delta t A\left(C_i\right) \frac{U\left(u_i^{n+2}\right) - U\left(u_i^n\right)}{2\Delta t} \varphi_i^n = \int_{\mathbb{R}^+} \int_{\mathbb{R}^2} U\left(u_h\right) \varphi_t + \mathcal{O}\left(h^{\frac{1}{2}}\right),$$

$$\text{d)} -\sum_{i,n} A(C_i)\left(U\left(u_i^{n+2}\right) - U\left(u_i^n\right)\right)\varphi_i^n$$
$$\geq -\int_{\mathbb{R}^+}\int_{\mathbb{R}^2} \mathbf{F}(u_h)\nabla\varphi + \mathcal{O}\left(h^{\frac{1}{2}}\right).$$

This is obvious as the left hand of a) and b) resp. c) and d) are identical.

Proof of a). For technical reasons, we extend φ on the domain $\mathbb{R}^2 \times \mathbb{R}$, such that $\varphi(\cdot, t) \equiv 0$ for $t > T$, φ and its derivatives still can be estimated by C_φ and $\varphi \in C_0^\infty\left(\mathbb{R}^2 \times \mathbb{R}\right)$. This is obviously possible.

Rewriting of the left hand by partial summation yields

$$\text{LH of a)} = -\sum_{i,n} 2\Delta t A(C_i)\, u_i^{n+2}\frac{\varphi_i^n - \varphi_i^{n+2}}{2\Delta t}$$
$$+\sum_i A(C_i)\, u_i^0\varphi_i^0 - \sum_i A(C_i)\, u_i^{N+2}\varphi_i^{N+2}.$$

The last term disappears as $\varphi(\cdot, (N+2)\Delta t) \equiv 0$. Thus for a) is sufficient to prove two relations:

$$(44)\quad \int_{\mathbb{R}^+}\int_{\mathbb{R}^2} u_h\varphi_t = \sum_{i,n} 2\Delta t A(C_i)\, u_i^{n+2}\frac{\varphi_i^{n+2} - \varphi_i^n}{2\Delta t} + \mathcal{O}\left(h^{\frac{1}{2}}\right)$$

$$(45)\quad \text{and} \int_{\mathbb{R}^2} u_0\varphi^0 = \sum_i A(C_i)\, u_i^0\varphi_i^0 + \mathcal{O}(h).$$

(45) follows from elementary calculations remembering the definition of the initial values (8), the Lipschitz-bound of φ and using that the integration-error of the midpoint-rule on polygons with diameter of $\mathcal{O}(h^2)$ has order of magnitude $\mathcal{O}(h^4)$.

Proof of (44): Starting with the left hand we successively cut off terms which are small, that means of magnitude $\mathcal{O}(h^{1/2})$. First we discretize the time integrals pointwise with error $R(x)$

$$\begin{aligned}\int_0^{\mathrm{T}}\int_{\mathbb{R}^2} u_h\varphi_t &= \sum_{i,j}\int_{L_{ij}\cap C_i}\int_0^{(N+2)\Delta t} u_h\varphi_t\\ &= \sum_{i,j}\int_{L_{ij}\cap C_i}\sum_n \Delta t\left(u_i^n\varphi_t^n + u_{ij}^{n+1}\varphi_t^{n+1}\right)\\ &\quad + \sum_{i,j}\int_{L_{ij}\cap C_i} R(x)\\ &=: T_5 + T_6.\end{aligned}$$

We obtain $T_6 = \mathcal{O}(h)$ as $|R(x)| \leq C_\varphi T \Delta t$ for all $x \in \mathbb{R}^2$. T_5 is rewritten as

$$T_5 = \sum_{i,j,n} \Delta t \int_{L_{ij}\cap C_i} 2u_i^n \varphi_t^n + \sum_{i,j,n} \Delta t \int_{L_{ij}\cap C_i} \left(u_{ij}^{n+1} \varphi_t^{n+1} - u_i^n \varphi_t^n \right)$$

(46) $=: T_7 + T_8.$

T_8 again is small, as further decomposition yields

$$\begin{aligned} T_8 &= \sum_{i,j,n} \Delta t \int_{L_{ij}\cap C_i} \left(u_{ij}^{n+1} - u_i^n \right) \varphi_t^n \\ &\quad + \sum_{i,j,n} \Delta t \int_{L_{ij}\cap C_i} u_{ij}^{n+1} \left(\varphi_t^{n+1} - \varphi_t^n \right) \\ &=: T_9 + T_{10}. \end{aligned}$$

First we obtain with (7), the CFL-condition (12) and $|u_{ij}^{n+1} - u_i^n| \leq |u_j^n - u_i^n|$ from the maximum-principle (15)

$$\begin{aligned} |T_9| &\leq \sum_{i,j,n} \Delta t \cdot A\left(L_{ij} \cap C_i\right) \cdot \left| u_{ij}^{n+1} - u_i^n \right| \cdot C_\varphi \\ &\leq \sum_{i,j,n} \frac{\beta h}{L} \cdot \frac{1}{3} b h^2 \cdot \left| u_j^n - u_i^n \right| \cdot C_\varphi = \frac{1}{3L} \beta b C_\varphi h \sum_{i,j,n} h^2 \left| u_j^n - u_i^n \right|, \end{aligned}$$

where we can apply the weak BV-estimate (39) and conclude

$$|T_9| \leq \frac{1}{3L} \beta b C_\varphi h \cdot C h^{-\frac{1}{2}} = \mathcal{O}\left(h^{\frac{1}{2}}\right).$$

Second we obtain

$$|T_{10}| \leq \sum_{i,j,n} \Delta t \cdot A\left(L_{ij} \cap C_i\right) \cdot B \cdot C_\varphi \cdot \Delta t = \mathcal{O}(h),$$

thus T_8 in (46) shown to be $\mathcal{O}(h^{1/2})$. The main term is T_7. With cellwise midpoint-integration we continue with error-terms R_i^n

$$\begin{aligned} T_7 &= \sum_{i,n} 2\Delta t u_i^n \int_{C_i} \varphi_t^n = \sum_{i,n} 2\Delta t u_i^n \varphi_t^n (S_i) A(C_i) + \sum_{i,n} 2\Delta t u_i^n R_i^n \\ &=: T_{11} + T_{12}. \end{aligned}$$

Every R_i^n is bounded by Ch^4 for a common constant C, therefore $T_{12} = \mathcal{O}\left(h^2\right)$. By Taylor-expansion we get $\varphi_t^n(S_i) = \left(\varphi_i^n - \varphi_i^{n-2}\right) / (2\Delta t) + \mathcal{O}(\Delta t)$. This yields

$$T_{11} = \sum_{i,n} 2\Delta t u_i^n A(C_i) \frac{\varphi_i^n - \varphi_i^{n-2}}{2\Delta t} + \mathcal{O}(\Delta t).$$

Let the dominating addend on the right hand of this equation be denoted by T_{13}. A reformulation of T_{13}, shows that it is equal to the main term of the right hand of (44) plus two small terms:

$$T_{13} = \sum_{i,n} 2\Delta t A(C_i) u_i^{n+2} \frac{\varphi_i^{n+2} - \varphi_i^n}{2\Delta t} + \sum_i 2\Delta t u_i^0 A(C_i) \frac{\varphi_i^0 - \varphi_i^{-2}}{2\Delta t}$$
$$- \sum_i 2\Delta t u_i^{N+2} A(C_i) \frac{\varphi_i^{N+2} - \varphi_i^N}{2\Delta t}.$$

The two last terms have order of magnitude $\mathcal{O}(h)$ because the fractions are bounded. This finishes the proof of (44) and herewith a).

Proof of b). It is sufficient to prove

$$\sum_{i,n} A(C_i)\left(u_i^{n+2} - u_i^n\right)\varphi_i^n$$
$$= \sum_{i,n} 2\Delta t \int_{C_i} \mathbf{f}(u_i^n) \nabla\varphi^n + \mathcal{O}\left(h^{\frac{1}{2}}\right) \tag{47}$$

and

$$\int_{\mathbb{R}+}\int_{\mathbb{R}^2} \mathbf{f}(u_h)\nabla\varphi = \sum_{i,n} 2\Delta t \int_{C_i} \mathbf{f}(u_i^n)\varphi^n + \mathcal{O}\left(h^{\frac{1}{2}}\right). \tag{48}$$

Equation (48) can be proven similar to (44). Starting with the left hand, again pointwise discretization of the time-integral is applied and the remaining terms are estimated using Lipschitz-bounds of $\mathbf{f}$, the discrete maximum principle and the weak BV-estimate.

Proof of (47): We first rewrite both sides. The left hand contains terms $A(C_i)(u_i^{n+2} - u_i^n)$. These are rewritten inserting both steps of the numerical scheme (9) and (10).

$$A(C_i)\left(u_i^{n+2} - u_i^n\right) \tag{49}$$
$$= A(C_i)\left(\sum_{j\in N_i}\left(r_{ij}u_{ij}^{n+1} - \frac{\Delta t}{A(C_i)}\mathbf{f}\left(u_{ij}^{n+1}\right)\boldsymbol{\theta}_{ij}\right) - u_i^n\right)$$
$$= \sum_{j\in N_i} r_{ij}A(C_i)\left(u_{ij}^{n+1} - u_i^n\right) - \Delta t \sum_{j\in N_i}\mathbf{f}\left(u_{ij}^{n+1}\right)\boldsymbol{\theta}_{ij} =: T_i^n - T_{i1}^n.$$

T_i^n can be further decomposed

$$T_i^n = \sum_{j\in N_i}\frac{1}{2}A(L_{ij})\left(\frac{u_j^n - u_i^n}{2} - \frac{\Delta t}{A(L_{ij})}\left(\mathbf{f}\left(u_j^n\right) - \mathbf{f}\left(u_i^n\right)\right)\boldsymbol{\theta}_{ij}\right)$$

$$= \sum_{j \in N_i} \frac{1}{4} A(L_{ij}) \left(u_j^n - u_i^n\right) - \Delta t \sum_{j \in N_i} \frac{1}{2} \left(\mathbf{f}\left(u_j^n\right) - \mathbf{f}\left(u_i^n\right)\right) \boldsymbol{\theta}_{ij}$$

$$=: T_{i2}^n - T_{i3}^n. \tag{50}$$

This decomposition $A(C_i)\left(u_i^{n+2} - u_i^n\right) = T_{i2}^n - T_{i3}^n - T_{i1}^n$ is inserted in the left hand of (47).

$$\begin{aligned} \text{LH of (47)} = & \sum_{i,j,n} \frac{1}{4} A(L_{ij}) \left(u_j^n - u_i^n\right) \varphi_i^n \\ & - \sum_{i,j,n} \frac{1}{2} \Delta t \left(\mathbf{f}\left(u_j^n\right) - \mathbf{f}\left(u_i^n\right)\right) \boldsymbol{\theta}_{ij} \varphi_i^n \\ & - \sum_{i,j,n} \Delta t \mathbf{f}\left(u_{ij}^{n+1}\right) \boldsymbol{\theta}_{ij} \varphi_i^n =: T_1 - T_2 - T_3. \end{aligned}$$

The dominating term on the right hand of (47) is rewritten with the divergence theorem and edgewise midpoint-integration. We use the notation φ_{ij}^{n+} (φ_{ij}^{n-}) for the evaluation of φ in the midpoint of the edge belonging to $\boldsymbol{\eta}_{ij}^+$ ($\boldsymbol{\eta}_{ij}^-$) at time $n\Delta t$, cf. Figures on page 465, and $\mathbf{n}$ denotes the outer normal of C_i.

$$\begin{aligned} \sum_{i,n} 2\Delta t \int_{C_i} \mathbf{f}\left(u_i^n\right) \nabla \varphi^n &= 2 \sum_{i,n} \Delta t \int_{\partial C_i} \left(\mathbf{f}\left(u_i^n\right) \varphi^n\right) d\mathbf{n} \\ &= 2 \sum_{i,n} \Delta t \sum_{j \in N_i} \mathbf{f}\left(u_i^n\right) \\ &\quad \times \left(\varphi_{ij}^{n-} \boldsymbol{\eta}_{ij}^- + \varphi_{ij}^{n+} \boldsymbol{\eta}_{ij}^+\right) + \mathcal{O}(h) \\ &=: 2T_4 + \mathcal{O}(h). \end{aligned}$$

With these notations the statement (47) reads

$$T_1 - T_2 - T_3 = 2T_4 + \mathcal{O}\left(h^{\frac{1}{2}}\right).$$

This will be proven in 3 steps:

$$T_1 = \mathcal{O}\left(h^{\frac{1}{2}}\right), \tag{51}$$

$$T_2 + T_4 = \mathcal{O}\left(h^{\frac{1}{2}}\right), \tag{52}$$

$$T_3 - T_2 = \mathcal{O}\left(h^{\frac{1}{2}}\right). \tag{53}$$

Proof of (51): T_1 can be made symmetric by splitting the sum, swapping i, j in one of the two parts and merging them again. This procedure is allowed although the summation range is not symmetric in i, j, as the boundary terms vanish due to the assumption on D. We obtain

$$T_1 = \sum_{i,j,n} \frac{1}{8} A\left(L_{ij}\right)\left(u_j^n - u_i^n\right)\left(\varphi_i^n - \varphi_j^n\right).$$

The Lipschitz-bound of φ, the area-bound (7) and the weak BV-estimate (39) yield exactly (51):

$$|T_1| \leq Ch \sum_{i,j,n} h^2 \left|u_j^n - u_i^n\right| = \mathcal{O}\left(h^{\frac{1}{2}}\right).$$

For (52) we rewrite T_2 with $\boldsymbol{\theta}_{ij} = \boldsymbol{\eta}_{ij}^+ + \boldsymbol{\eta}_{ij}^-$, T_4 is symmetrized as described above. We obtain the representations

$$T_2 = \sum_{i,j,n} \frac{1}{2} \Delta t \left(\mathbf{f}\left(u_j^n\right) - \mathbf{f}\left(u_i^n\right)\right)\left(\boldsymbol{\eta}_{ij}^+ \varphi_i^n + \boldsymbol{\eta}_{ij}^- \varphi_i^n\right),$$

$$T_4 = \sum_{i,j,n} \frac{1}{2} \Delta t \left(\mathbf{f}\left(u_i^n\right) - \mathbf{f}\left(u_j^n\right)\right)\left(\varphi_{ij}^{n-} \boldsymbol{\eta}_{ij}^- + \varphi_{ij}^{n+} \boldsymbol{\eta}_{ij}^+\right),$$

as $\varphi_{ij}^{n-} \boldsymbol{\eta}_{ij}^- + \varphi_{ij}^{n+} \boldsymbol{\eta}_{ij}^+ = -\varphi_{ji}^{n-} \boldsymbol{\eta}_{ji}^- - \varphi_{ji}^{n+} \boldsymbol{\eta}_{ji}^+$. This yields

$$\begin{aligned} T_2 + T_4 = \sum_{i,j,n} & \frac{1}{2} \Delta t \left(\mathbf{f}\left(u_j^n\right) - \mathbf{f}\left(u_i^n\right)\right) \\ & \times \left(\boldsymbol{\eta}_{ij}^+ \left(\varphi_i^n - \varphi_{ij}^{n+}\right) + \boldsymbol{\eta}_{ij}^- \left(\varphi_i^n - \varphi_{ij}^{n-}\right)\right). \end{aligned} \tag{54}$$

The absolute values of the single differences appearing in this term can be estimated using Lipschitz-properties of φ and $\mathbf{f}$. With 3.5 b) and the weak BV-estimate we conclude with statement (52)

$$|T_2 + T_4| \leq C \Delta t \sum_{i,j,n} \left|u_j^n - u_i^n\right| \cdot h^2 = \mathcal{O}\left(h^{\frac{1}{2}}\right).$$

Proof of (53): First T_2 is rewritten by changing the sign of the second flux contribution. This is allowed as this term vanishes in the sum over j.

$$T_2 = \sum_{i,j,n} \frac{1}{2} \Delta t \left(\mathbf{f}\left(u_j^n\right) + \mathbf{f}\left(u_i^n\right)\right) \boldsymbol{\theta}_{ij} \varphi_i^n,$$

We subtract T_3 and T_2, the result is made symmetric resulting in

$$\begin{aligned} & T_3 - T_2 \\ & = \sum_{i,j,n} \Delta t \frac{1}{2}\left(-\mathbf{f}\left(u_i^n\right) + 2\mathbf{f}\left(u_{ij}^{n+1}\right) - \mathbf{f}\left(u_j^n\right)\right) \varphi_i^n \boldsymbol{\theta}_{ij} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,n} \frac{1}{4} \Delta t \left(-\mathbf{f}\left(u_i^n\right) + 2\mathbf{f}\left(u_{ij}^{n+1}\right) - \mathbf{f}\left(u_j^n\right) \right) \left(\varphi_i^n - \varphi_j^n\right) \boldsymbol{\theta}_{ij} \\
&= \sum_{i,j,n} \frac{1}{4} \Delta t \left(\mathbf{f}\left(u_{ij}^{n+1}\right) - \mathbf{f}\left(u_j^n\right) \right) \left(\varphi_i^n - \varphi_j^n\right) \boldsymbol{\theta}_{ij} \\
&\quad + \sum_{i,j,n} \Delta t \frac{1}{4} \left(\mathbf{f}\left(u_{ij}^{n+1}\right) - \mathbf{f}\left(u_i^n\right) \right) \left(\varphi_i^n - \varphi_j^n\right) \boldsymbol{\theta}_{ij} \ =: T_5 + T_6.
\end{aligned}
$$

Both sums can be estimated similar to (54) additionally using the discrete maximum-principle. Both terms T_5 and T_6 turn out to have order of magnitude $\mathcal{O}(h^{1/2})$, thus the last relation (53) is shown to be valid, herewith (47).

c) and d) can be proven in a similar manner since their structure agrees with the structure of a) resp. b). This finishes the proof of Proposition 4.10.

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References

1. Arminjon, P., Madrane, A., St. Cyr, A. Numerical simulation of 3-D Flows with a non-oscillatory central scheme on unstructured tetrahedral grids. Talk presented at the Eighth International Conference on Hyperbolic Problems, Magdeburg 2000
2. Arminjon, P., Stanescu D., Viallon M.C., A two-dimensional finite volume extension of the Lax-Friedrichs and Nessyahu-Tadmor schemes for compressible flows, Proceedings of the 6th Int. Symposium on Comp. Fluid Dynamics, September 4–8, 1995, Lake Tahoe, **4**, 7–14
3. Arminjon, P., Viallon, M.-C., Généralisation du schéma de Nessyahu-Tadmor pour une équation hyperbolique à deux dimensions d'espace. C. R. Acad. Sci. Paris, **320**, Série 1, 85–88 (1995)
4. Arminjon, P., Viallon, M.-C., Madrane, A. A Finite Volume Extension of the Lax-Friedrichs and Nessyahu-Tadmor Schemes for Conservation Laws on unstructured Grids. IJCFD, **9**, 1–22 (1997)
5. Arminjon, P., Viallon, M.-C., Convergence of a finite volume extension of the Nessayhu-Tadmor scheme on unstructured grids for a two-dimensional linear hyperbolic equation. SIAM J. Numer. Anal. **36**, (3) 738–771 (1999)
6. Bouchut, F., Perthame, B., Kruzkov's estimates for scalar conservation laws revisited. Trans. Am. Math. Soc. **350**, (7) 2847–2870 (1998)
7. Cockburn, B., Coquel, F., LeFloch, P., Convergence of the finite volume method for multidimensional conservation laws. SIAM J. Numer. Anal. **2**, (3) 687–705 (1995)
8. DiPerna, R.J. Measure-valued solutions to conservation laws. Arch. Ration. Mech. Anal. **88**, 223–270 (1985)
9. Jiang, G.-S., Levy, D., Lin, C.-T., Osher, S., Tadmor, E., High-resolution nonoscillatory central schemes with nonstaggered grids for hyperbolic conservation laws. SIAM J. Numer. Anal. **35**, (6) 2147–2168 (1998)
10. Jiang, G.-S., Tadmor, E., Non-oscillatory central schemes for multidimensional hyperbolic conservation laws. SIAM J. Sci. Comput. **19**, (6) 1892–1917 (1998)

11. Kröner, D., Rokyta, M., Convergence of upwind finite volume schemes for scalar conservation laws in two dimensions. SIAM J. Numer. Anal. **31**, (2) 324–343 (1994)
12. Kröner, D., Noelle, S., Rokyta, M., Convergence of higher order upwind finite volume schemes on unstructured grids for scalar conservation laws in several space dimensions. Numer. Math. **71**, (4) 527–560 (1995)
13. Kurganov, A., Tadmor, E., New High-Resolution Central Schemes for Nonlinear Conservation Laws and Convection-Diffusion Equations. Preprint, April 1999
14. Levy, D., Tadmor, E., Non-oscillatory central schemes for the incompressible 2-D Euler equations. Math. Res. Lett. **4**, (2–3) 321–340 (1997)
15. Liu, X.-D., Tadmor, E., Third order nonoscillatory central scheme for hyperbolic conservation laws. Numer. Math. **79**, (3) 397–425 (1998)
16. Nessyahu, H., Tadmor, E., Non-oscillatory central differencing for hyperbolic conservation laws. J. Comp. Physics, **87**, (2) 408–463 (1990)
17. Rohde, C., Upwind finite volume schemes for weakly coupled hyperbolic systems of conservation laws in 2D. Numer. Math. **81**, (1) 85–124 (1998)
18. Tartar, L., Compensated compactness and applications to partial differential equations. Research notes in mathematics, nonlinear analysis and mechanics: Heriot-Watt Symposium, **4**, 136–212 (1979)
19. Tartar, L., The compensated compactness method applied to systems of conservation laws. Systems of nonlinear partial differential equations. NATO ASI Ser. C **111**, 263–285 (1983)