D'Alembert and Lagrange

E. Gekeler

12/06/09

Consider a mechanical system with position vector $\underline{x}(t) \in \mathbb{R}^n$ obeying NEWTON's law

$$C\ddot{\underline{x}} - \underline{f}(\underline{x}) = \underline{0} \tag{1}$$

where C is a symmetric, positive definite matrix and $\underline{f}$ is the *impressed force*.

The left side of (1) can no longer be equal to zero in case there are constraint qualifications for the orbit of $\underline{x}$:

Suppose, e.g., that there are m holonomic-scleronomic constraint conditions

$$\underline{g}(\underline{x}) = \underline{0} \in \mathbb{R}^m\,, \quad \mathrm{rank}\,\mathrm{grad}\,\underline{g} = m\,. \tag{2}$$

The set $\mathcal{M} = \{\underline{x} \in \mathbb{R}^n\,,\ \underline{g}(\underline{x}) = \underline{0}\}$ forms an m-dimensional nonlinear manifold in $\mathbb{R}^n$. By derivation for the independent variable t we obtain $\mathrm{grad}\,\underline{g}(\underline{x})\dot{\underline{x}} = 0$ under the assumption that the orbit $\underline{x}$ satisfies the constraints. Therefore the m *rows* of $\mathrm{grad}\,\underline{g}$ are (unnormed) normals to $\mathcal{M}$ and span the normal space of $\mathcal{M}$ (written as *column* vectors); they are the *constraint forces* introduced by LAGRANGE. Also $n - m$ linear independent vectors $\delta\underline{x}_i \in \mathbb{R}^n$ (notation historically) satisfying $\mathrm{grad}\,g(\underline{x})\delta\underline{x}_i = \underline{0}$ are the tangents to the manifold $\mathcal{M}$ and span the tangent space of $\mathcal{M}$; they are the *virtual displacements* of D'ALEMBERT. The LAGRANGE multiplier method works with the normals of the manifold $\mathcal{M}$ and D'ALEMBERT's principle works with the tangents to the manifold $\mathcal{M}$.

By an ingenious idea of LAGRANGE, a linear combination of constraint forces is added to the impressed force in (1). In other words, the zero right side is replaced by $y_1[\mathrm{grad}\,g_1(\underline{x})]^T + \ldots + y_m[\mathrm{grad}\,g_m(\underline{x})]^T \equiv [\mathrm{grad}\,\underline{g}(\underline{x})]^T\underline{y} \in \mathbb{R}^n$ such that

$$C\ddot{\underline{x}} - \underline{f}(\underline{x}) = [\mathrm{grad}\,\underline{g}(\underline{x})]^T\underline{y} \in \mathbb{R}^n\,, \quad \underline{g}(\underline{x}) = \underline{0} \in \mathbb{R}^m\,. \tag{3}$$

The components y_i of that linear combination are the LAGRANGE multipliers or *costate variables*. The augmented system (3) is a differential-algebraic system and is solved for $\underline{x}$ and $\underline{y}$.

Unlike today, solving such systems was not popular in ancient times although $\ddot{\underline{x}}$ appears explicitly in the differential system of (3). To circumvent this problem the (older) D'ALEMBERT has proposed to solve the n-dimensional system

$$[C\ddot{\underline{x}} - \underline{f}(\underline{x})]^T\delta\underline{x}_i(\underline{x}) = 0\,,\ i = 1 : n - m\,,\ \underline{g}(\underline{x}) = \underline{0} \in \mathbb{R}^m \tag{4}$$

for the n-dimensional unknown $\underline{x}$. This means that the left side of (1) must be perpendicular to the tangent space of $\mathcal{M}$. In other words, the transposed system (3) is multiplied by the tangents $\delta\underline{x}_i$, $i = 1 : n - m$ such that the constraint forces (and the multipliers y_i) disappear ("constraint forces do not perform real work"). Of course, (4) is also a differential-algebraic system but without additional variables. By a skilful elimination of surplus variables a *pure* differential system can be found in less complicated cases, at least in those cases where $\underline{g}$ is affin linear in $\underline{x}$.

Both devices, LAGRANGE's multiplier method and D'ALEMBERT's principle are geometrically equivalent by a simple result of Linear Algebra (Range Theorem).

For further details and examples see, e.g.,

E.Gekeler: Mathematical Methods for Mechanics, Springer-Verlag 2008.