Supplements 1 to Chapter X

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On the Christoffel symbols

$$\boxed{\Gamma_{ij}^{k} = \frac{1}{2} g^{kl} (g_{il|j} + g_{jl|i} + g_{ij|l})}\ , \tag{1}$$

Proof.

$$\begin{aligned} g_{il|j} &= \Gamma_{ij}^{k} g_{kl} + \Gamma_{lj}^{k} g_{ik} \\ g_{jl|i} &= \Gamma_{ji}^{k} g_{kl} + \Gamma_{li}^{k} g_{jk} \\ g_{ij|l} &= \Gamma_{il}^{k} g_{kj} + \Gamma_{jl}^{k} g_{ik} \end{aligned}$$

Multiplication by g^{kl} yields

$$\begin{aligned} g^{kl} g_{il|j} &= \Gamma_{ij}^{k} + g^{kl} \Gamma_{lj}^{k} g_{ik} \\ g^{kl} g_{jl|i} &= \Gamma_{ji}^{k} + g^{kl} \Gamma_{li}^{k} g_{jk} \end{aligned}$$

Addition yields because of symmetry

$$2\,\Gamma_{ij}^{k} = g^{kl} \Big[g_{il|j} + g_{jl|i} + \Gamma_{lj}^{k} g_{ik} + \Gamma_{li}^{k} g_{jk} \Big]$$

$$\Gamma_{lj}^{k} g_{ik} + \Gamma_{li}^{k} g_{jk} = \Gamma_{li}^{k} g_{kj} + \Gamma_{jl}^{k} g_{ik} = g_{ij|l}$$