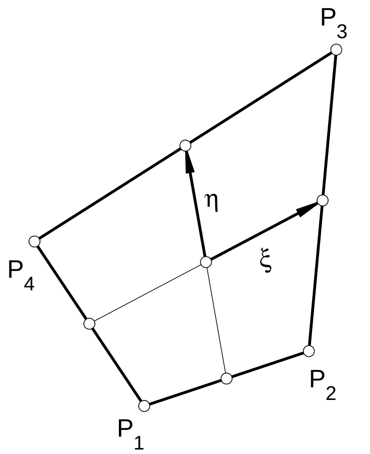
Supplements 4 to Chapter IX

E. Gekeler

11/10/06

Case Study: Rannacher-Turek element

Non-conform (convex) quadrangle element, the nodes are the mid-points of the edges.



Shape functions in unit square and in the special local coordinate system:

$$1\,,\ \xi\,,\ \eta\,,\ \xi^2-\eta^2\,.$$

By reasons of stability (!), a special local coordinate system is introduce int the quadrangle, s. sketch.

Lineare transformation into global coordinates:

$$x=\alpha_1+\beta_1\xi+\gamma_1\eta\,,\quad y=\alpha_2+\beta_2\xi+\gamma_2\eta\,.$$

$$\begin{array}{rclrcl}
\alpha_1 &=& (x_1+x_2+x_3+x_4)/4, & \alpha_2 &=& (y_1+y_2+y_3+y_4)/4\\
\beta_1 &=& (x_2+x_3-x_1-x_4)/4, & \beta_2 &=& (y_2+y_3-y_1-y_4)/4,\\
\gamma_1 &=& (x_3+x_4-x_1-x_2)/4, & \gamma_2 &=& (y_3+y_4-y_1-y_2)/4
\end{array}$$

Design matrix

$$B=\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1\\ 0 & 2 & 0 & -2\\ -2 & 0 & 2 & 0\\ -1 & 1 & -1 & 1\end{bmatrix}$$

Shape functions in special local coordinate system

$$\begin{array}{rclrcl}
\Psi_1(\xi,\eta) &=& [(1-\eta)^2-\xi^2]/4 & \Psi_2(\xi,\eta) &=& [(1+\xi)^2-\eta^2]/4\\
\Psi_3(\xi,\eta) &=& [(1+\eta)^2-\xi^2]/4 & \Psi_4(\xi,\eta) &=& [(1-\xi)^2-\eta^2]/4
\end{array}$$

Global-local transformation for substitution

$$\begin{bmatrix}\xi\\ \eta\end{bmatrix}=\frac{1}{\beta_1\gamma_2-\beta_2\gamma_1}\begin{bmatrix}\gamma_2 & -\gamma_1\\ -\beta_2 & \beta_2\end{bmatrix}\begin{bmatrix}x-\alpha_1\\ y-\alpha_2\end{bmatrix}$$

At first the shape functions must be written in global coordinates and the derivatives w.r.t. x and y are to be calculated. To apply GAUSS integration, the quadrangle is divided into two triangles, and then, after a further substitution, GAUSS rules in unit triangle are applied.