Supplements 3 to Chapter IX

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Case Study: Complete Cubic Approach

$$\begin{aligned} v(\xi,\eta) &= a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 \\ &= +a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3 \\ v_\xi(\xi,\eta) &= a_2 + 2a_4\xi + a_5\eta + 3a_7\xi^2 + 2a_8\xi\eta + a_9\eta^2 \\ v_\eta(\xi,\eta) &= a_3 + a_5\xi + 2a_6\eta + a_8\xi^2 + a_9 2\xi\eta + 3a_{10}\eta^2 \end{aligned}$$

(a) Starting from the algebraic basis

$$\Theta(\xi,\eta) = [1,\, \xi,\, \eta,\, \xi^2,\, \xi\eta,\, \eta^2,\, \xi^3,\, \xi^2\eta,\, \xi\eta^2,\, \eta^3]T$$

the basis $\Psi = [\psi_1, \ldots, \psi_1 0]$ of shape functions in unit triangle is calculated by means of the design matrix B:

$$\Psi(\xi,\eta) = \Theta(\xi,\eta)B\,. \qquad (1)$$

Succession for the point in unit triangle and the center:

$$Q_1(0,0) : \psi_1,\psi_2,\psi_3,\; Q_2(1,0) : \psi_4,\psi_5,\psi_6,$$
$$Q_3(0,1) : \psi_7,\psi_8,\psi_9,\; Q_4(1/3,1/3) : \psi_{10}.$$

$$\begin{aligned}
\psi_1 &= (1-\xi-\eta)[(1-\xi+2\eta)(1+2\xi-\eta) - 16\xi\eta] &&= \zeta_1^2(3-2\zeta_1) - 7\zeta_1\zeta_2\zeta_3 \\
\psi_2 &= \xi(1-\xi-2\eta)(1-\xi-\eta) &&= \zeta_1\zeta_2(\zeta_1-\zeta_3) \\
\psi_3 &= \eta(1-2\xi-\eta)(1-\xi-\eta) &&= \zeta_1\zeta_3(\zeta_1-\zeta_2) \\
\psi_4 &= \xi^2(3-2\xi) - 7\xi\eta(1-\xi-\eta) &&= \zeta_2^2(3-2\zeta_2) - 7\zeta_1\zeta_2\zeta_3 \\
\psi_5 &= \xi^2(\xi-1) + 2\xi\eta(1-\xi-\eta) &&= \zeta_2^2(\zeta_2-1) + 2\zeta_1\zeta_2\zeta_3 \\
\psi_6 &= -\xi\eta(1-2x-\eta) &&= -\zeta_2\zeta_3(\zeta_1-\zeta_2) \\
\psi_7 &= \eta^2(3-2\eta) - 7\xi\eta(1-\xi-\eta) &&= \zeta_3^2(3-2\zeta_3) - 7\zeta_1\zeta_2\zeta_3 \\
\psi_8 &= -\xi\eta(1-\xi-2\eta) &&= -\zeta_2\zeta_3(\zeta_1-\zeta_3) \\
\psi_9 &= \eta^2(\eta-1) + 2\xi\eta(1-\xi-\eta) &&= \zeta_3^2(\zeta_3-1) + 2\zeta_1\zeta_2\zeta_3 \\
\psi_{10} &= 27\xi\eta(1-\xi-\eta) &&= 27\zeta_1\zeta_2\zeta_3\,.
\end{aligned}$$

Now we calculate the following matrices in unit triangle

$$\begin{aligned} S_1 &= \int_S \Psi_\xi\Psi_\xi^T\, d\xi d\eta\,, \quad S_2 = \int_S [\Psi_\xi\Psi_\eta^T + \Psi_\eta\Psi_\xi^T]\, d\xi d\eta \\ S_3 &= \int_S \Psi_\eta\Psi_\eta^T\, d\xi d\eta\,, \quad S_4 = \int_S \Psi\Psi^T\, d\xi d\eta \end{aligned}$$

In triangle $T(x,y)$ then with shape functions

$$\Phi(x,y) = [\varphi_1(x,y), \ldots, \varphi_{10}(x,y)]^T$$

$$\begin{aligned} a(u,v) &= \int_T \operatorname{grad} u \cdot \operatorname{grad} v\, dxdy = \int_T [u_x v_x + u_y v_y]\, dxdy \\ &= U^T\left(\int_T \Phi_x\Phi_x^T\, dxdy + \int_T \Phi_y\Phi_y^T\, dxdy\right)V \end{aligned}$$

where $U,\, V \in \mathbb{R}^{10}$ are the local node vectors.