Supplements 2 to Chapter IX

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Case Study: Mini element

Cubic bubble function in unit triangle

$$\beta(\xi,\eta) = 17\,\xi\eta(1-\xi-\eta)\,,\ \beta_\xi = 27\eta(1-2\xi-\eta)\,,\ \beta_\eta = 27\xi(1-\xi-2\eta)$$

Augmented linear ansatz for velocity component

$$\begin{aligned} v(\xi,\eta) &= a_1 + a_2\xi + a_3\eta + a_4\beta(\xi,\eta) \\ v_\xi(\xi,\eta) &= a_2 + a_4\beta_\xi(\xi,\eta) \\ v_\eta(\xi,\eta) &= a_3 + a_4\beta_\eta(\xi,\eta) \end{aligned}$$

Algebraic basis

$$\begin{aligned} \Theta(\xi,\eta) &= [1,\ \xi,\ \eta,\ \beta(\xi,\eta)]T \\ \Theta_\xi &= [0,\ 1,\ 0,\ \beta_\xi]^T \\ \Theta_\eta &= [0,\ 0,\ 1,\ \beta_\eta]^T \end{aligned}$$

Basis $\Psi = [\psi_1,\dots,\psi_4]$ of shape functions in unit triangle with succession $(0,0),(1,0),(0,1),(1/3,1/3)$

$$\begin{aligned} \psi_1 &= (1-\xi-\eta) &&= \zeta_3 \\ \psi_2 &= \xi &&= \zeta_1 \\ \psi_3 &= \eta &&= \zeta_2 \\ \psi_4 &= 27\xi\eta(1-\xi-\eta) &&= \zeta_1\zeta_2\zeta_3 \end{aligned}$$

For the ordinary Mini-element we have $\underline{u} = A\underline{a}$ with the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1/3 & 1/3 & 1 \end{bmatrix}\,,\ B = A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ -3 & 0 & 3 & 0 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

Let, with the algebraic basis

$$\widetilde{S}_1 = \int_S \Theta_\xi\Theta_\xi^T\,d\xi d\eta\,,\ \widetilde{S}_2 = \int_S \Theta_\xi\Theta_\eta^T\,d\xi d\eta\,,\ \widetilde{S}_3 = \int_S \Theta_\eta\Theta_\eta^T\,d\xi d\eta\,,\ \widetilde{S}_4 = \int_S \Theta\Theta^T\,d\xi d\eta$$

and (regard S_2)

$$\begin{aligned} S_1 &= B^T\widetilde{S}_1B &&= \int_S \Psi_\xi\Psi_\xi^T\,d\xi d\eta \\ S_2 &= B^T(\widetilde{S}_2+\widetilde{S}_2^T)B &&= \int_S [\Psi_\xi\Psi_\eta^T + \Psi_\eta\Psi_\xi^T]\,d\xi d\eta \\ S_3 &= B^T\widetilde{S}_3B &&= \int_S \Psi_\eta\Psi_\eta^T\,d\xi d\eta \\ S_4 &= B^T\widetilde{S}_4B &&= \int_S \Psi\Psi^T\,d\xi d\eta \end{aligned}$$

then

$$S_1 = \frac{1}{6}\begin{bmatrix} 3 & 1 & 0 & -4 & 0 & 0 \\ 1 & 3 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 0 & -8 & 8 \end{bmatrix}\,,\quad S_2 = \frac{1}{6}\begin{bmatrix} 6 & 1 & 1 & -4 & 0 & -4 \\ 1 & 0 & -1 & -4 & 4 & 0 \\ 1 & -1 & 0 & 0 & 4 & -4 \\ -4 & -4 & 0 & 8 & -8 & 8 \\ 0 & 4 & 4 & -8 & 8 & -8 \\ -4 & 0 & -4 & 8 & -8 & 8 \end{bmatrix}\,,$$

$$S_3 = \frac{1}{6}\begin{bmatrix} 3 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & -4 \\ 0 & 0 & 0 & 8 & -8 & 0 \\ 0 & 0 & 0 & -8 & 8 & 0 \\ -4 & 0 & -4 & 0 & 0 & 8 \end{bmatrix}, \quad S_4 = \frac{1}{360}\begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix},$$

Further, we have the representation

$$\Psi_\xi = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 27 & -2*27 & -27 \end{bmatrix}\begin{bmatrix} 1 \\ \xi \\ \eta \\ \xi\eta \\ \eta^2 \end{bmatrix}, \quad \Psi_\eta = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 27 & -27 & -2*27 \end{bmatrix}\begin{bmatrix} 1 \\ \xi \\ \eta \\ \xi\eta \\ \eta^2 \end{bmatrix}. \quad (1)$$

For the linear shape functions of the pressure we have

$$\widetilde{\Psi}(\xi,\eta) = \begin{bmatrix} 1-\xi-\eta \\ \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix}$$

$\widetilde{\Theta} = [1,\xi,\eta]^T$, $|S| = 1/2$,

$$M = \int_S \widetilde{\Theta}\widetilde{\Theta}^T \, d\xi d\eta = \frac{1}{24}\begin{bmatrix} 12 & 4 & 4 \\ 4 & 2 & 1 \\ 4 & 1 & 2 \end{bmatrix}$$

$$C_1 = \int_S \Psi_\xi \widetilde{\Psi}^T \, d\xi d\eta = \frac{1}{6}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{bmatrix}, \quad C_2 = \int_S \Psi_\eta \widetilde{\Psi}^T \, d\xi d\eta = \frac{1}{6}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Lineare Stokes equation in triangle T

$$\begin{aligned} a(\underline{u},\underline{v}) &= \int_T \operatorname{grad}\underline{u} : \operatorname{grad}\underline{v}\, dxdy = \int_T [u_{1,x}v_{1,x} + u_{1,y}v_{1,y}]\, dxdy + \int_T [u_{2,x}v_{2,x} + u_{2,y}v_{2,y}]\, dxdy \\ &= \widetilde{a}(u_1,v_1) + \widetilde{a}(u_2,v_2) \\ b(\underline{u},q) &= \int_T \operatorname{div}\underline{u}\cdot q\, dxdy = \int_T (u_{1,x} + u_{2,y})\, q\, dxdy \\ &= \int_T (u_{1,\xi}\xi_x + u_{1,\eta}\eta_x)\, q\, dxdy + \int_T (u_{2,\xi}\xi_y + u_{2,\eta}\eta_y)\, q\, dxdy \end{aligned}$$

Further,

$$b(\underline{v},q) = \int_T \operatorname{div}\underline{v}\cdot q\, dxdy = -\int \operatorname{grad} q\cdot \underline{v}\, dxdy$$

$$\begin{aligned} a(\underline{v},\underline{u}) \; - \; (\operatorname{div}\underline{v},p) &= (\underline{v},\underline{f}) + (\underline{v},\underline{\sigma}_n(\underline{u},p))_\Gamma\,, \quad \underline{v}\in\mathcal{V} \\ -(\operatorname{div}\underline{v},q) &= 0\,, \qquad q\in\mathcal{Q} \\ \underline{\sigma}_n(\underline{u},p) &= (2\nu\varepsilon(\underline{u}) - p)\underline{n} \end{aligned}$$

Trilinear Form with shape functions.
$P(\underline{u},\underline{v},\underline{w}) = \underline{u}^T[\text{grad}\,\underline{v}]\underline{w}$

$$\underline{u} \simeq \begin{bmatrix} \Phi(x,y)^T U_1 \\ \Phi(x,y)^T U_2 \end{bmatrix},$$

$$N(\underline{u},\underline{v},\underline{w}) \simeq \ [U_1^T, U_2^T]\begin{bmatrix} (\Phi_x^T V_1)\Phi\,\Phi^T & (\Phi_y^T V_1)\Phi\,\Phi^T \\ (\Phi_x^T V_2)\Phi\,\Phi^T & (\Phi_y^T V_2)\Phi\,\Phi^T \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

$$\Phi_x = \Psi_\xi \xi_x + \Psi_\eta \eta_x\,, \quad \Phi_y = \Psi_\xi \xi_y + \Psi_\eta \eta_y\,,$$

$$\int_T N(\underline{u},\underline{v},\underline{w}) \simeq [U_1^T, U_2^T]\begin{bmatrix} A(V_1) & B(V_1) \\ C(V_2) & D(V_2) \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

where

$$\begin{aligned} A(V_1) &= J\int_S \big((\Psi_\xi \xi_x + \Psi_\eta \eta_x)^T V_1\big)\Psi\,\Psi^T\,d\xi d\eta\,, \\ B(V_1) &= J\int_S \big((\Psi_\xi \xi_y + \Psi_\eta \eta_y)^T V_1\big)\Psi\,\Psi^T\,d\xi d\eta \\ C(V_2) &= J\int_S \big((\Psi_\xi \xi_x + \Psi_\eta \eta_x) V_2\big)\Psi\,\Psi^T\,d\xi d\eta\,, \\ D(V_2) &= J\int_S \big((\Psi_\xi \xi_y + \Psi_\eta \eta_y) V_2\big)\Psi\,\Psi^T\,d\xi d\eta \end{aligned}$$

Thus we have to calculate

$$A_i = \int_S \psi_{i,\xi}\Psi\,\Psi^T\,d\xi d\eta\,, \quad B_i = \int_S \psi_{i,\eta}\Psi\,\Psi^T\,d\xi d\eta\,,\ i = 1:n\,.$$

But the functions $\psi_{i,\xi}$, $\psi_{i,\eta}$ are additive composed by the functions 1 , ξ , η, therefore only the both matrices P , Q are to be calculated besides the mass matrix:

$$M = \int_S \Psi\,\Psi^T\,d\xi d\eta\,, \quad P = \int_S \xi\Psi\,\Psi^T\,d\xi d\eta\,, \quad Q = \int_S \eta\Psi\,\Psi^T\,d\xi d\eta\,,$$

as well as, by (1),

$$\begin{array}{lll} A_1 = -3M + 4P + 4Q\,, & A_2 = -M + 4P\,, & A_3 = 0\,, \\ A_4 = 4M - 8P - 4Q\,, & A_5 = 4Q\,, & A_6 = -4Q \\ B_1 = -3M + 4P + 4Q\,, & B_2 = 0\,, & B_3 = -M + 4Q\,, \\ B_4 = -4P\,, & B_5 = 4P\,, & B_6 = 4M - 4P - 4Q \end{array}$$

$$P_1 = \sum_{i=1}^6 A_i v_{1,i}\,, \quad P_2 = \sum_{i=1}^6 B_i v_{1,i}\,, \quad P_3 = \sum_{i=1}^6 A_i v_{2,i}\,, \quad P_4 = \sum_{i=1}^6 B_i v_{2,i}\,.$$

$$A = J(P_1\xi_x + P_2\eta_x)\,, \quad B = J(P_1\xi_y + P_2\eta_y)\,, \quad C = J(P_3\xi_x + P_4\eta_x)\,, \quad D = J(P_3\xi_y + P_4\eta_y)\,.$$