Supplements 1 to Chapter IX

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Case Study: Taylor-Hood element

Complete quadratic “ansatz” for each of the velocity components and linear ansatz for the pressure component in unit triangle S:

$$\begin{aligned} v(\xi,\eta) &= a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 \\ q(\xi,\eta) &= b_1 + b_2\xi + b_3\eta \end{aligned}$$

(a) Starting from the algebraic basis

$$\Theta(\xi,\eta) = [1,\ \xi,\ \eta,\ \xi^2,\ \xi\eta,\ \eta^2]T$$

the basis $\Psi = [\psi_1,\dots,\psi_6]$ of shape functions in unit triangle is calculated by means of the design matrix B:

$$\Psi(\xi,\eta) = \Theta(\xi,\eta)B\,. \tag{1}$$

in successione $(0,0),(1,0),(0,1),(1/2,0),(1/2,1/2),(0,1/2)$

$$\begin{aligned} u_1 = v(0,0) &= a_1 \\ u_1 = v(1,0) &= a_1 + a_2 + a_4 \\ u_3 = v(0,1) &= a_1 + a_3 + a_5 \\ u_4 = v(1/2,0) &= a_1 + a_2/2 + a_4/4 \\ u_5 = v(1/2,1/2) &= a_1 + a_2/2 + a_3/2 + a_4/4 + a_5/4 + a_6/4 \\ u_6 = v(0,1/2) &= a_1 + a_3/2 + a_6/4 \end{aligned}$$

It follows $\underline{u} = A\underline{a},\ \underline{a} = B\underline{u}$ where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1/2 & 0 & 1/4 & 0 & 0 \\ 1 & 1/2 & 1/2 & 1/4 & 1/4 & 1/4 \\ 1 & 0 & 1/2 & 0 & 0 & 1/4 \end{bmatrix},\ B = A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 4 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 4 \\ 2 & 2 & 0 & -4 & 0 & 0 \\ 4 & 0 & 0 & -4 & 4 & -4 \\ 2 & 0 & 2 & 0 & 0 & -4 \end{bmatrix}.$$

and then, by (1),

$$\begin{aligned} \psi_1 &= (1-\xi-\eta)(1-2\xi-2\eta) &&= \zeta_1(2\zeta_1 - 1) \\ \psi_2 &= \xi(2\xi-1) &&= \zeta_2(2\zeta_2-1) \\ \psi_3 &= \eta(2\eta-1) &&= \zeta_3(2\zeta_3-1) \\ \psi_4 &= 4\xi(1-\xi-\eta) &&= 4\zeta_1\zeta_2 \\ \psi_5 &= 4\xi\eta &&= 4\zeta_2\zeta_3 \\ \psi_6 &= 4\eta(1-\xi-\eta) &&= 4\zeta_1\zeta_3 \end{aligned}$$

Now we calculate the following matrices in unit triangle

$$\begin{aligned} S_1 &= \int_S \Psi_\xi\Psi_\xi^T\, d\xi d\eta\,, & S_2 &= \int_S [\Psi_\xi\Psi_\eta^T + \Psi_\eta\Psi_\xi^T]\, d\xi d\eta \\ S_3 &= \int_S \Psi_\eta\Psi_\eta^T\, d\xi d\eta\,, & S_4 &= \int_S \Psi\Psi^T\, d\xi d\eta \end{aligned}$$

then

$$S_1 = \frac{1}{6}\begin{bmatrix} 3 & 1 & 0 & -4 & 0 & 0 \\ 1 & 3 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -4 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & -8 \\ 0 & 0 & 0 & 0 & -8 & 8 \end{bmatrix}, \quad S_2 = \frac{1}{6}\begin{bmatrix} 6 & 1 & 1 & -4 & 0 & -4 \\ 1 & 0 & -1 & -4 & 4 & 0 \\ 1 & -1 & 0 & 0 & 4 & -4 \\ -4 & -4 & 0 & 8 & -8 & 8 \\ 0 & 4 & 4 & -8 & 8 & -8 \\ -4 & 0 & -4 & 8 & -8 & 8 \end{bmatrix},$$

$$S_3 = \frac{1}{6}\begin{bmatrix} 3 & 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 & -4 \\ 0 & 0 & 0 & 8 & -8 & 0 \\ 0 & 0 & 0 & -8 & 8 & 0 \\ -4 & 0 & -4 & 0 & 0 & 8 \end{bmatrix}, \quad S_4 = \frac{1}{360}\begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ -1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix},$$

Likewise one calculates the shape functions for linear approximation in unit triangle:

$$\widetilde{\Psi} = [1-\xi-\eta\,,\ \xi\,,\eta]^T$$

and the both integrals

$$C_1 = \int_S \Psi_\xi \widetilde{\Psi}^T\, d\xi d\eta = \frac{1}{6}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \\ -1 & -1 & -2 \end{bmatrix}, \quad C_2 = \int_S \Psi_\eta \widetilde{\Psi}^T\, d\xi d\eta = \frac{1}{6}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

and

$$S_5 = \int_S \Psi \widetilde{\Psi}^T\, d\xi d\eta = \frac{1}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Stationary Navier-Stokes equations with convection term:

$$\begin{aligned} a(\underline{v},\underline{u}) + c(\underline{v},\underline{u},\underline{u}) \quad - \quad (\operatorname{div}\underline{v}, p) &= (\underline{v},\underline{f}) + (\underline{v},\underline{\sigma}_n(\underline{u},p))_\Gamma\,, \quad \underline{v}\in\mathcal{V} \\ -(\operatorname{div}\underline{v}, q) &= 0\,, \qquad q\in\mathcal{Q} \\ \underline{\sigma}_n(\underline{u},p) &= (2\nu\varepsilon(\underline{u}) - p)\underline{n} \end{aligned}$$

In arbitrary triangle $T(x,y)$ we have

$$\begin{aligned} a(\underline{u},\underline{v}) &= \nu\int_T \operatorname{grad}\underline{u} : \operatorname{grad}\underline{v}\, dxdy \\ &= \nu\int_T [u_{1,x}v_{1,x} + u_{1,y}v_{1,y}]\, dxdy + \nu\int_T [u_{2,x}v_{2,x} + u_{2,y}v_{2,y}]\, dxdy \\ &= \widetilde{a}(u_1,v_1) + \widetilde{a}(u_2,v_2) \\ b(\underline{u},q) &= \int_T \operatorname{div}\underline{u}\cdot q\, dxdy = \int_T (u_{1,x} + u_{2,y})\, q\, dxdy \\ c(\underline{u},\underline{v},\underline{w}) &= \int_T \underline{u}\cdot(\operatorname{grad}\underline{v})\underline{w}\, dxdy \end{aligned}$$

For transformation on the unit triangle we have e.g.

$$\begin{aligned} b(\underline{u}, q) &= \int_T (u_{1,\xi}\xi_x + u_{1,\eta}\eta_x)\, q\, dxdy + \int_T (u_{2,\xi}\xi_y + u_{2,\eta}\eta_y)\, q\, dxdy \\ &= J\int_S (u_{1,\xi}\xi_x + u_{1,\eta}\eta_x)\, q\, d\xi d\eta + J\int_S (u_{2,\xi}\xi_y + u_{2,\eta}\eta_y)\, q\, d\xi d\eta \\ &= \frac{J}{J} U_1^T (C_1 y_{31} - C_2 y_{21}) Q + \frac{J}{J} U_2^T (C_2 x_{21} - C_1 x_{31}) Q\,, \end{aligned}$$

where $U_1, U_2 \in \mathbb{R}^6\,,\ Q \in \mathbb{R}^3$ are the local vectors of the node variables in the triangle
(b) Trilinear Form with shape functions.
$P(\underline{u}, \underline{v}, \underline{w}) = \underline{u}^T[\text{grad}\,\underline{v}]\underline{w}$

$$\underline{u} \simeq \begin{bmatrix} \Phi(x,y)^T U_1 \\ \Phi(x,y)^T U_2 \end{bmatrix},$$

$$N(\underline{u}, \underline{v}, \underline{w}) \simeq\ [U_1^T, U_2^T] \begin{bmatrix} (\Phi_x^T V_1)\Phi\,\Phi^T & (\Phi_y^T V_1)\Phi\,\Phi^T \\ (\Phi_x^T V_2)\Phi\,\Phi^T & (\Phi_y^T V_2)\Phi\,\Phi^T \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

$$\Phi_x = \Psi_\xi \xi_x + \Psi_\eta \eta_x\,,\quad \Phi_y = \Psi_\xi \xi_y + \Psi_\eta \eta_y\,,$$

$$\int_T N(\underline{u}, \underline{v}, \underline{w}) \simeq [U_1^T, U_2^T] \begin{bmatrix} A(V_1) & B(V_1) \\ C(V_2) & D(V_2) \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix},$$

where

$$\begin{aligned} A(V_1) &= J\int_S \big((\Psi_\xi \xi_x + \Psi_\eta \eta_x)^T V_1\big)\Psi\,\Psi^T\, d\xi d\eta\,, \\ B(V_1) &= J\int_S \big((\Psi_\xi \xi_y + \Psi_\eta \eta_y)^T V_1\big)\Psi\,\Psi^T\, d\xi d\eta \\ C(V_2) &= J\int_S \big((\Psi_\xi \xi_x + \Psi_\eta \eta_x) V_2\big)\Psi\,\Psi^T\, d\xi d\eta\,, \\ D(V_2) &= J\int_S \big((\Psi_\xi \xi_y + \Psi_\eta \eta_y) V_2\big)\Psi\,\Psi^T\, d\xi d\eta \end{aligned} \tag{2}$$

Now we have the representation

$$\Psi_\xi = \begin{bmatrix} -3 & 4 & 4 \\ -1 & 4 & 0 \\ 0 & 0 & 0 \\ 4 & -8 & -4 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix} =: [A_1\,,\ A_2\,,\ A_3] \begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix}$$

$$\Psi_\eta = \begin{bmatrix} -3 & 4 & 4 \\ 0 & 0 & 0 \\ -1 & 0 & 4 \\ 0 & -4 & 0 \\ 0 & 4 & 0 \\ 4 & -4 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix} =: [B_1\,,\ B_2\,,\ B_3] \begin{bmatrix} 1 \\ \xi \\ \eta \end{bmatrix}$$

Accordingly, only the both matrices $P\,, Q$ are to be calculated:

$$M := S_4 = \int_S \Psi\,\Psi^T\, d\xi d\eta\,,\quad P = \int_S \xi\Psi\,\Psi^T\, d\xi d\eta\,,\quad Q = \int_S \eta\Psi\,\Psi^T\, d\xi d\eta\,,$$

Then we obtain

$$\begin{aligned}\int_S (\Psi_\xi V)\Psi\Psi^T\, d\xi d\eta &= (A_1^T V)M + (A_2^T V)P + (A_3^T V)Q\\ \int_S (\Psi_\eta V)\Psi\Psi^T\, d\xi d\eta &= (B_1^T V)M + (B_2^T V)P + (B_3^T V)Q\end{aligned}$$

and the integrals (2) are linear combinations of these both integrals with the corresponding arguments V_1 resp. V_2

$$\begin{aligned}A(V_1) &= [(A_1 y_{31} - B_1 y_{21})V_1]M + [(A_2 y_{31} - B_2 y_{21})V_1]P + [A_3 y_{31} - B_3 y_{21})V_1]Q\\ B(V_1) &= [(-A_1 x_{31} + B_1 x_{21})V_1]M + [(-A_2 x_{31} + B_2 x_{21})V_1]P + [(-A_3 x_{31} + B_3 x_{21})V_1]Q\\ C(V_2) &= [(A_1 y_{31} - B_1 y_{21})V_2]M + [(A_2 y_{31} - B_2 y_{21})V_2]P + [A_3 y_{31} - B_3 y_{21})V_2]Q\\ D(V_2) &= [(-A_1 x_{31} + B_1 x_{21})V_2]M + [(-A_2 x_{31} + B_2 x_{21})V_2]P + [(-A_3 x_{31} + B_3 x_{21})V_2]Q\end{aligned}$$

(c) The nonlinear problem may be solved by a simple iteration method or by Newton's method. The latter allows higher Reynolds numbers $Re = 1/\nu$. Also, in Newton's method the gradient of the nonlinear part must be calculated. From **(b)**

$$\operatorname{grad}_{U_1,U_2}\begin{bmatrix} A(U_1) & B(U_1)\\ C(U_2) & D(U_2)\end{bmatrix}\begin{bmatrix} U_1\\ U_2\end{bmatrix} = \begin{bmatrix} A(U_1) & B(U_1)\\ C(U_2) & D(U_2)\end{bmatrix} + \begin{bmatrix} A(U_1) & \text{Null}\\ \text{Null} & D(U_2)\end{bmatrix}$$

(d) Postprozessor Let $\underline{u} = (u, v)$ be the velocity field then the streamlines z satisfy

$$\int_\Omega \nabla\delta z\, \nabla z = \int_\Omega \delta z\, w + \int_\Gamma \delta z\, z_{\underline{n}}$$

$$\int_T \varphi_i w = \int_T \varphi_i (v_x - u_y) + \int_\Gamma \delta z\, z_{\underline{n}}$$

$$\begin{aligned}&\int_T \delta z\, w\, dxdy\\ &\simeq JZ \int_S \Psi(\Psi_\xi \xi_x + \Psi_\eta \eta_x)^T\, d\xi d\eta V - JZ \int_S \Psi(\Psi_\xi \xi_y + \Psi_\eta \eta_y)^T\, d\xi d\eta U\\ &= JZ \left[\int_S \Psi\Psi_\xi^T\, d\xi d\eta\right](V\xi_x - U\xi_y) + JZ \left[\int_S \Psi\Psi_\eta^T\, d\xi d\eta\right](V\eta_x - U\eta_y)\end{aligned}$$

Accorcingly, the matrices

$$\int_S \Psi\Psi_\xi^T\, d\xi d\eta\,, \quad \int_S \Psi\Psi_\eta^T\, d\xi d\eta$$

are to be calculated.