Supplements 1 to Chapter VIII

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Derivation of the Differential Form of the Balance Law of Angular Momentum
Let $x = (x^1, x^2, x^3)^T$, let $\underline{t}^i$ be the rows of the stress tensor $\underline{t}$, and let

$$\underline{c} = \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix} = \begin{bmatrix} \underline{c}^1 \\ \underline{c}^2 \\ \underline{c}^3 \end{bmatrix}.$$

Then

$$x \times \underline{t}\,\underline{n} = \begin{bmatrix} x^2\underline{t}^3 - x^3\underline{t}^2 \\ x^3\underline{t}^1 - x^1\underline{t}^3 \\ x^1\underline{t}^2 - x^2\underline{t}^1 \end{bmatrix} \underline{n} = \underline{c}\,\underline{t}\,\underline{n},$$

and the product rule

$$\operatorname{div}(\underline{c} \cdot \underline{t}) = [\operatorname{grad}(\underline{c}^{i,T}) : \underline{t}]_{i=1}^3 + \underline{c}\, \operatorname{div} \underline{t}$$

as well as the formula

$$[\operatorname{grad}(\underline{c}^{i,T}) : \underline{t}]_{i=1}^3 = \begin{bmatrix} t_{32} - t_{23} \\ t_{13} - t_{31} \\ t_{21} - t_{12} \end{bmatrix} =: \underline{t}^*.$$

Applying the divergence theorem row by row, yields

$$\begin{aligned} &\int_{\partial\Phi(U,t)} x \times \underline{t}\,\underline{n}\, do = \int_{\partial\Phi(U,t)} \underline{c}\,\underline{t}\,\underline{n}\, do \\ &= \int_{\Phi(U,t)} \operatorname{div}(\underline{c} \cdot \underline{t})\, dv = \int_{\Phi(U,t)} [\underline{t}^* + \underline{c}\, \operatorname{div} \underline{t}]\, dv. \end{aligned}$$

Let $\underline{c}\, \operatorname{div} \underline{t} = x \times \operatorname{div} \underline{t}$ then we obtain for the law of angular momentum

$$\begin{aligned} &\frac{D}{Dt} \int_{\Phi(U,t)} \varrho(x,t)[x \times \underline{v}(x,t)]\, dv = \int_{\Phi(U,t)} \varrho(x,t) \frac{D}{Dt}[x \times \underline{v}(x,t)]\, dv \\ &= \int_{\Phi(U,t)} [x \times \underline{f}(x,t)]\, dv + \int_{\partial\Phi(U,t)} x \times \underline{t}(x,t)\underline{n}(x,t)\, do \\ &= \int_{\Phi(U,t)} [x \times \underline{f} + \underline{t}^* + x \times \operatorname{div} \underline{t}]\, dv. \end{aligned} \tag{1}$$

But, recalling

$$\frac{Dx}{Dt} \times \underline{v} = \underline{v} \times \underline{v} = 0$$

and the law of momentum in differential form

$$\varrho \frac{D\underline{v}}{Dt} = \operatorname{div} \underline{t} + \underline{f},$$

we obtain by (8.13)

$$\begin{aligned} &\int_{\Phi(U,t)} \varrho \frac{D}{Dt}(x \times \underline{v})\, dv = \int_{\Phi(U,t)} \varrho \left[\frac{Dx}{Dt} \times \underline{v} + x \times \frac{D\underline{v}}{Dt}\right] dv \\ &= \int_{\Phi(U,t)} x \times \varrho \frac{D\underline{v}}{Dt}\, dv = \int_{\Phi(U,t)} [x \times (\operatorname{div}\, \underline{t} + \underline{f})]\, dv \end{aligned}$$

A comparison with (1) ergibt

$$\int_{\Phi(U,t)} \underline{t}^* \, dv = 0.$$

This result must be valid for all subvolumes U and thus yields the differential form of the balance law of angular momentum as $\underline{t}^* = 0$ which is equivalent with $\underline{t} = \underline{t}^T$. Accordingly, this balance law is already be given by the symmetry of the stress tensor $\underline{t}$.