Supplements 2 to Chapter VI

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Examples to the D'Alembert-Lagrange Principle

In the following examples with pendulum the x-axis points to right and the y-axis to above (KOS). *Example 1:* The simple mathematical pendulum has two variables $\underline{x} = [x_1, x_2]^T$, $(x_2 = y_1)$, one constraint $x_1^2 + x_2^2 = \ell^2$ and the total energy

$$\begin{aligned} E(t) &= \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2) + mgx_2 = E(t_0) \\ \frac{dE(t)}{dt} &= [m\ddot{x}_1, m\ddot{x}_2 + mg] \cdot \underline{\dot{x}}(t) =: \widetilde{\text{grad}} E(t) \cdot \underline{x}(t) = 0\,. \end{aligned}$$

The tangent $\underline{t} = [x_2, -x_1]^T$ stands perpendicular on the normal of the curve $\underline{n} = [x_1, x_2]^T$, and we have $\text{grad}\, E = \underline{z}$ with the constraint force $\underline{z} = \lambda \underline{n}$. The scalar product $(\text{grad}\, E - \lambda \underline{n}) \cdot \underline{t} = 0$ yields

$$m(\ddot{x}_1 x_2 - \ddot{x}_2 x_1) - mgx_1 = 0\,. \tag{1}$$

Let φ be the angle between the *negative* y-axis and the point (x_1, x_2) as position vector:

$$\begin{aligned} x_1 = \ell \sin\varphi\,,\ \dot{x}_1 = \ell \cos\varphi\dot{\varphi}\,,\ \ddot{x}_1 = -\ell \sin\varphi(\dot{\varphi})^2 + \ell\cos\varphi\ddot{\varphi} \\ x_2 = -\ell\cos\varphi\,,\ \dot{x}_2 = \ell\sin\varphi\dot{\varphi}\,,\ \ddot{x}_2 = \ell\cos\varphi(\dot{\varphi})^2 + \ell\sin\varphi\ddot{\varphi} \end{aligned}$$

$$\ddot{x}_1 x_2 - \ddot{x}_2 x_1 = -\ell^2 \ddot{\varphi}$$

Inserting into (1) yields

$$\boxed{-m\ell^2\ddot{\varphi} - mg\ell\sin\varphi = 0}\,.$$

Example 2: The simple physical pendulum (body pendulum) has three variables $\underline{x}(t) = [x_1, x_2, \varphi]^T$, two consztraints

$$x_1 - \ell\sin\varphi = 0\,,\quad x_2 + \ell\cos\varphi = 0\,, \tag{2}$$

and the total energy

$$\begin{aligned} E(t) &= \frac{1}{2}\Theta\dot{\varphi}^2 + Mgx_2 = E(t_0) \\ \frac{dE(t)}{dt} &= [0, Mg, \Theta\ddot{\varphi}]\, \underline{\dot{x}} =: \widetilde{\text{grad}} E \cdot \underline{\dot{x}} = 0 \end{aligned}$$

(Θ inertia moment relative to rotational axis, M total mass, (x_1, x_2) coordinates of the gravity center). Two normal vectors are $\underline{n}_1 = [1, 0, -\ell\cos\varphi]^T$, $\underline{n}_2 = [0, 1, -\ell\sin\varphi]^T$ and a tangent $\underline{t}$ is obtained as cross product $\underline{t} = \underline{n}_1 \times \underline{n}_2 = [\ell\cos\varphi, \ell\sin\varphi, 1]^T$. The scalar product $\widetilde{\text{grad}} E \cdot \underline{t} = 0$ yields

$$\boxed{[0, Mg, \Theta\ddot{\varphi}] \begin{bmatrix} \ell\cos\varphi, \\ \ell\sin\varphi \\ 1 \end{bmatrix} = M\ell g\sin\varphi + \Theta\ddot{\varphi} = 0}\,.$$

Example 3: The double physical pendulum has six variables $\underline{x} = [x_1, \ldots, x_4, \varphi_1, \varphi_2]^T$ and four constraint (see figure)

$$x_1 - \ell_1\sin\varphi_1 = 0\,,\ x_2 + \ell_1\cos\varphi_1 = 0\,,\ x_3 - \ell\sin\varphi_1 - \ell_2\sin\varphi_2 = 0\,,\ x_4 + \ell\cos\varphi_1 + \ell_2\cos\varphi_2 = 0\,. \tag{3}$$

(x_1, y_1) are the coordinates of the gravity center S_1 and the inertia moment of the first body relative to the rotational axis is Θ_1, (x_3, x_4) are the coordinates of the second gravity center S_2 and Θ_2 is the corresponding inertia moment w.r.t. S_2.(!) Kinetic energy T, potential energy U and total energy $E = T + U$ are

$$\begin{aligned}
T(t) &= \frac{1}{2}\Theta_1\dot{\varphi}_1^2 + \frac{1}{2}\Theta_2\dot{\varphi}_2^2 + \frac{1}{2}m_2(\dot{x}_3^2 + \dot{x}_4^2) \\
U(t) &= m_1 g x_2 + m_2 g x_4 \\
E(t) &= \frac{1}{2}\Theta_1\dot{\varphi}_1^2 + \frac{1}{2}\Theta_2\dot{\varphi}_2^2 + \frac{1}{2}m_2(\dot{x}_3^2 + \dot{x}_4^2) + m_1 g x_2 + m_2 g x_4 = E(t_0) \qquad (4) \\
\frac{dE(t)}{dt} &= [0,\, m_1 g,\, m_2\ddot{x}_3,\, m_2\ddot{x}_4 + m_2 g,\, \Theta_1\ddot{\varphi}_1,\, \Theta_2\ddot{\varphi}_2] \cdot \dot{\underline{x}} \\
&\widetilde{\mathrm{grad}}E \cdot \dot{\underline{x}} = 0\,.
\end{aligned}$$

The both desired tangents $\underline{t}_1$, $\underline{t}_2$ are two linearly independent solutions of the homogeneous linear system

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\ell_1\cos\varphi_1 & 0 \\ 0 & 1 & 0 & 0 & -\ell_1\sin\varphi_1 & 0 \\ 0 & 0 & 1 & 0 & -\ell\cos\varphi_1 & -\ell_2\cos\varphi_2 \\ 0 & 0 & 0 & 1 & -\ell\sin\varphi_1 & -\ell_2\sin\varphi_2 \end{bmatrix} \underline{t} = 0$$

hence

$$\begin{bmatrix} \underline{t}_1^T \\ \underline{t}_2^T \end{bmatrix} = \begin{bmatrix} \ell_1\cos\varphi_1 & \ell_1\sin\varphi_1 & \ell\cos\varphi_1 & \ell\sin\varphi_1 & 1 & 0 \\ 0 & 0 & \ell_2\cos\varphi_2 & \ell_2\sin\varphi_2 & 0 & 1 \end{bmatrix}.$$

The both equations $\widetilde{\mathrm{grad}}E \cdot \underline{t}_1 = 0$ and $\widetilde{\mathrm{grad}}E \cdot \underline{t}_2 = 0$ yields

$$\begin{aligned}
&m_1\ell_1 g\sin\varphi_1 + m_2\ell\ddot{x}_3\cos\varphi_1 + m_2\ell\sin\varphi_1(\ddot{x}_4 + g) + \Theta_1\ddot{\varphi}_1 = 0 \\
&m_2\ell_2\ddot{x}_3\cos\varphi_2 + m_2\ell_2\sin\varphi_2(\ddot{x}_4 + g) + \Theta_2\ddot{\varphi}_2 = 0
\end{aligned}$$

or

$$\begin{aligned}
\Theta_1\ddot{\varphi}_1 \;+\; \ell m_2(\cos\varphi_1\ddot{x}_3 + \sin\varphi_1\ddot{x}_4) &= -g(\ell_1 m_1 + \ell m_2)\sin\varphi_1 \\
\Theta_2\ddot{\varphi}_2 \;+\; \ell_2 m_2(\cos\varphi_2\ddot{x}_3 + \sin\varphi_2\ddot{x}_4) &= -\ell_2 m_2 g\sin\varphi_2
\end{aligned}$$

We have

$$\begin{aligned}
\ddot{x}_3 &= -\ell\sin\varphi_1(\dot{\varphi}_1)^2 - \ell_2\sin\varphi_2(\dot{\varphi}_2)^2 + \ell\cos\varphi_1\ddot{\varphi}_1 + \ell_2\cos\varphi_2\ddot{\varphi}_2 \\
\ddot{x}_4 &= \ell\cos\varphi_1(\dot{\varphi}_1)^2 + \ell_2\cos\varphi_2(\dot{\varphi}_2)^2 + \ell\sin\varphi_1\ddot{\varphi}_1 + \ell_2\sin\varphi_2\ddot{\varphi}_2
\end{aligned}$$

$$\begin{aligned}
\cos\varphi_1\ddot{x}_3 + \sin\varphi_1\ddot{x}_4 &= \ell\ddot{\varphi}_1 + \ell_2\cos(\varphi_1 - \varphi_2)\ddot{\varphi}_2 + \ell_2\sin(\varphi_1 - \varphi_2)(\dot{\varphi}_2)^2 \\
\cos\varphi_2\ddot{x}_3 + \sin\varphi_2\ddot{x}_4 &= \ell_2\ddot{\varphi}_2 + \ell\cos(\varphi_1 - \varphi_2)\ddot{\varphi}_1 + \ell\sin(\varphi_2 - \varphi_1)(\dot{\varphi}_1)^2
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\Theta_1 + \ell^2 m_2)\ddot{\varphi}_1 + \ell\ell_2 m_2\cos(\varphi_1 - \varphi_2)\ddot{\varphi}_2 + \ell\ell_2 m_2\sin(\varphi_1 - \varphi_2)(\dot{\varphi}_2)^2 \\
&= -(\ell_1 m_1 + \ell m_2)g\sin\varphi_1 \\
&(\Theta_2 + \ell_2^2 m_2)\ddot{\varphi}_2 + \ell\ell_2 m_2\cos(\varphi_1 - \varphi_2)\ddot{\varphi}_1 - \ell\ell_2 m_2\sin(\varphi_1 - \varphi_2)(\dot{\varphi}_1)^2 \\
&= -\ell_2 m_2 g\sin\varphi_2
\end{aligned}$$

$$
\begin{aligned}
& \begin{bmatrix} \Theta_1 + \ell^2 m_2 & \ell\ell_2 m_2 \cos(\varphi_1 - \varphi_2) \\ \ell\ell_2 m_2 \cos(\varphi_1 - \varphi_2) & \Theta_2 + \ell_2^2 m_2 \end{bmatrix} \begin{bmatrix} \ddot{\varphi}_1 \\ \ddot{\varphi}_2 \end{bmatrix} \\
+ \quad & \ell\ell_2 m_2 \begin{bmatrix} \sin(\varphi_1 - \varphi_2)(\dot{\varphi}_2)^2 \\ -\sin(\varphi_1 - \varphi_2)(\dot{\varphi}_1)^2 \end{bmatrix} \\
= \quad & -\begin{bmatrix} (\ell_1 m_1 + \ell m_2) g \sin\varphi_1 \\ \ell_2 m_2 g \sin\varphi_2 \end{bmatrix}
\end{aligned}
$$

The same result is also found by Hamilton's principle and the Euler equations; cf. Szabo: Höhere TM, S. 89.

Example with holonomic-rheonomic constraint, [Hamel], S. 328. A point with mass $m = 1$ moves in the (x, y)-plane accorcing to

$$U = -\frac{1}{2} a(t) x^2 \,, \quad y = c(t) x \,.$$

Total energy

$$
\begin{aligned}
E(t) &= \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2} a(t) x(t)^2 = E(t_0) \\
\frac{dE(t)}{dt} &= \ddot{x} - a(t)x)\dot{x} + \ddot{y}\dot{y} - \frac{1}{2}\dot{a}(t) x^2 = 0 \qquad (5)
\end{aligned}
$$

Total derivation of the constraint w.r.t. t yields

$$\dot{y} = c\dot{x} + \dot{c}x \implies [\dot{x},\, \dot{y}] = [\dot{x},\, c\dot{x} + \dot{c}x]$$

Inserting into (5) and elimination of y by means of the constraint yields

$$(\ddot{x} - ax)\dot{x} + (\ddot{x}c + 2\dot{x}\dot{c} + x\ddot{c})(c\dot{x} + x\dot{c}) - \frac{1}{2}\dot{a}x^2 = 0 \,. \qquad (6)$$

According to [Hamel] a solution exists

$$x(t) = \frac{\alpha t + \beta}{(1 + c^2)^{1/2}} \,, \quad \text{for} \quad a(t) = \frac{\dot{c}^2}{1 + c^2} \,.$$

This solution satisfies (6).