Supplements 1 to Chapter VI

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11/10/06

Den Herrn D'ALEMBERT halte ich für einen großen mathematicum in abstractis; aber wenn er einen incursum macht in mathesin applicatam, so höret alles estime bei mir auf: sein Hydronamica ist viel zu kindisch, daß ich einige estime für ihn in dergleichen Sachen haben könnte.
D.BERNOULLI 1750 in einem Brief an Euler

Examples to virtual work

Axiom 1 *(Principle of minimal potential energy) A conservative mechanical system is in equilibrium if the total potential energy takes a stationary value.*

In simple cases this principle has some advantages against the equivalent *principle of virtual work* since only the gradient w.r.t. the *independent* variables must be set equal to zero in order to obtain a necessary and in many cases also sufficient condition for a solution.

Example 1. (Cf. [Gross I], I,§8.3.) A weightless rod of length ℓ is hinged at the one end A (= origin) and a force $\underline{k}$ atacks at the other end. Which position takes the rod?

Solution: The point B moves on a circle $\underline{x} = (x, y) = \ell(\cos\varphi, \sin\varphi)$. We have for potential energy

$$E(\varphi) = \underline{k} \cdot \underline{x}(\varphi)\,.$$

$dE(\varphi)/d\varphi = 0$ is a necessary condition for a stationary point hence

$$\underline{k} \cdot \underline{x}'(\varphi) = 0, \quad \underline{x}'(\varphi) = \ell[-\sin\varphi,\, \cos\varphi]^T\,,$$

or $\tan\varphi = k_2/k_1$ (but this is already known).

Example 2. (Drawbridge, cf. [Gross a], I, §8.3.) A rod of length ℓ and weight G is hinged at the point A (= origin). The other end is hold by a weightless rope and the weight $Q(0, y)$ where the rope runs over a roll R$(0, \ell)$. Which angle φ against the x-axis is attained? For $\varphi = \pi/2$ let $y = 0$.

Solution: Let $S(\frac{\ell}{2}(\cos\varphi, \sin\varphi))$ be the gravity center then the total energy is

$$E(y, \varphi) = Q \cdot y + \frac{G \cdot \ell}{2} \sin\varphi\,.$$

Further

$$\frac{y}{2\ell} = \sin\left(\frac{\pi/2 - \varphi}{2}\right) \Longleftrightarrow y = 2\ell\left(\frac{1 - \cos(\frac{\pi}{2} - \varphi)}{2}\right)^{1/2} \Longleftrightarrow y = \sqrt{2}\ell(1 - \sin\varphi)^{1/2}\,.$$

therefore

$$F(\varphi) := 2E(\varphi)/\ell = 2\sqrt{2}Q(1 - \sin\varphi)^{1/2} + G\sin\varphi\,.$$

Setting the derivation equal to zero yields

$$\sqrt{2}Q\frac{-\cos\varphi}{(1 - \sin\varphi)^{1/2}} + G\cos\varphi = 0\,.$$

This yields $\varphi = \pi/2$ or

$$\boxed{\sin\varphi = \frac{G^2 - 2Q^2}{G^2}} \Longrightarrow \sin\left(\frac{\frac{\pi}{2} - \varphi}{2}\right) = \frac{Q}{G}\,.$$

Example 3. (Cf. [Gross a], I, § 8.3.) Which moment must attack at a (frictionless) spindle with slope h that a weight G is hold?

Solution: Total energy

$$E(\varphi) = M \cdot \varphi - G\frac{h}{2\pi}\varphi\,,$$

hence $M = G\frac{h}{2\pi}$.

Example 4. (Cf. [Gross a], I, § 8.3. Example 8.3) Two rods S_1, S_2 with terminal points A, B resp. B, C are pin-jointed in B. A is the origin in which S_1 is hinged. Weight and length of S_1 resp. S_2 are G_1, L_1 bzw. G_2, L_2. Position at rest is $-\pi/2$ for S_1 and for S_2. Which equilibrium position is taken if a force $\underline{k}$ attacks in C?

Solution: The trajectories of B and C satisfy

$$\begin{array}{lrcl}
B: & \underline{u}(\varphi) & = & \big(L_1\cos(-\frac{\pi}{2}+\varphi),\ L_1\sin(-\frac{\pi}{2}+\varphi)\big) = (L_1\sin\varphi,\ -L_1\cos\varphi),\\
 & \underline{u}_\varphi(\varphi) & = & (L_1\cos\varphi,\ L_1\sin\varphi),\\
C: & \underline{v}(\varphi,\psi) & = & \big(L_1\cos(-\frac{\pi}{2}+\varphi)+L_2\cos(-\frac{\pi}{2}+\psi),\ L_1\sin(-\frac{\pi}{2}+\varphi)+L_2\sin(-\frac{\pi}{2}+\psi)\big)\\
 & & = & (L_1\sin\varphi+L_2\sin\psi,\ -L_1\cos\varphi-L_2\cos\psi)\\
 & \frac{d}{d\varphi}\underline{v}(\varphi,\psi) & = & (L_1\cos\varphi,\ L_1\sin\varphi)\\
 & \frac{d}{d\psi}\underline{v}(\varphi,\psi) & = & (L_2\cos\psi,\ L_2\sin\psi)\,.
\end{array}$$

Therefore the system has two degrees of freedom. For instance, the rod S_1 behaves like a point of mass G_1 at distance $L_1/2$ of A. Then there follows for the energy

$$E(\varphi,\psi) = \underline{k}\cdot\underline{v}(\varphi,\,\psi) + \begin{bmatrix}0\\-G_1\end{bmatrix}\cdot\frac{1}{2}\underline{u}(\varphi) + \begin{bmatrix}0\\-G_2\end{bmatrix}\cdot\big(\frac{1}{2}\underline{v}(\varphi,\,\psi)+\underline{u}(\varphi)\big)\,.$$

Setting the derivation equal to zero yields

$$\begin{array}{rcl}
E_\varphi(\varphi,\psi) & = & \underline{k}\cdot\underline{v}_\varphi(\varphi,\,\psi) + \begin{bmatrix}0\\-G_1\end{bmatrix}\cdot\frac{1}{2}\underline{u}_\varphi(\varphi) + \begin{bmatrix}0\\-G_2\end{bmatrix}\cdot\big(\frac{1}{2}\underline{v}_\varphi(\varphi,\,\psi)+\underline{u}_\varphi(\varphi)\big) = 0\\
E_\psi(\varphi,\psi) & = & \underline{k}\cdot\underline{v}_\psi(\varphi,\,\psi) + \begin{bmatrix}0\\-G_2\end{bmatrix}\cdot\frac{1}{2}\underline{v}_\psi(\varphi,\,\psi) = 0
\end{array}$$

or

$$\begin{array}{l}
k_1L_1\cos\varphi + k_2L_1\sin\varphi - \frac{G_1}{2}L_1\sin\varphi - G_2L_1\sin\varphi\\
= k_1L_1\cos\varphi + \left(k_2L_1 - \frac{G_1+2G_2}{2}L_1\right)\sin\varphi = 0\\
k_1L_2\cos\psi + k_2L_2\sin\psi - G_2L_2\sin\psi\\
= k_1L_2\cos\psi + \left(k_2L_2 - \frac{G_2}{2}L_2\right)\sin\psi = 0
\end{array}$$

or

$$\boxed{\tan\varphi = \frac{2k_1}{G_1+2G_2-2k_2}\,,\quad \tan\psi = \frac{2k_1}{G_2-2k_2}}\,.$$