Supplements 5 to Chapter V

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The Perturbed Eigenvalue Problem Let $\mathcal{E} \subset \mathcal{F}$ be *complex* Banach spaces, let $0 \in \mathcal{I} \subset \mathbb{R}$ be an open interval and $L(\,\cdot\,) : \mathcal{I} \ni \varepsilon \mapsto L(\varepsilon) \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ an operator-valued function; in particular $\mathcal{E} = \mathcal{F} = \mathbb{C}^n$ may be the complex coordinate space. We consider the mapping

$$\boxed{F : \mathcal{I} \times (\mathbb{C} \times \mathcal{E}) \ni (\varepsilon, (\lambda, x)) \to L(\varepsilon)x - \lambda\, x = 0 \in \mathcal{F}}\,, \tag{1}$$

where now (λ, x) plays the role of x and ε the role of μ in § 5.4 **(c)**. Possibly after some translation, the point $\varepsilon = 0$ is examined for branching. For a general overview on the perturbed eigenvalue problem it is referred to [Chow], chap. 14, and the references there. Here we suppose that $L(\varepsilon)$ is a non-selfadjoint Fredholm operator with index zero and derive a necessary and sufficient condition for the smoothness of characteristic pairs emanating from a semi-simple eigenvalue of $L(0)$. Without restriction, we choose this eigenvalue to be zero and confine ourselves to the case of a complex Banach space $\mathcal{F}$. Concerning the finite-dimensional case, some hints to our result are found in [Golub], p. 204, and also in [Wikinson] (1965), chap. II, §17, 25.
More exactly, we make the following assumption on L:

Assumption 1 (1°) $L := L(0)$ *is a* Fredholm *operator with index zero.*

$$r = \dim \operatorname{Ker}(L) = \dim \operatorname{Ker}(L_d),$$
$$\operatorname{Ker}(L) = \operatorname{span}\{u_1, \ldots, u_r\}\,,\ \ \operatorname{Ker}(L_d) = \operatorname{span}\{v^1, \ldots, v^r\}.$$

(2°) $\mathcal{E} \subset \mathcal{F}$ *and* $\operatorname{Ker} L \cap \operatorname{Range} L = \{0\}$.
(3°) L *is continuously differentiable in* $\mathcal{I}$ *such that*

$$L(\varepsilon) = L + \varepsilon B + o(|\varepsilon|)\,,\quad |\varepsilon| \to 0\,,$$

where $B = L_\varepsilon^0 := L_\varepsilon(0) \in \mathcal{L}(\mathcal{E}, \mathcal{F})$.

We use again the formal matrices $U = [u_1, \ldots, u_r]$, $V = [v^1, \ldots, v^r]$, the formal notation $U\xi = \sum_{i=1}^r u_i \xi^i$, and write $(v, u) := v(u)$ for $u \in \mathcal{E}$ and $v \in \mathcal{E}_d$. Then we have e.g.

$$V^d B U = [(v^i, Bu_j)]_{i,j=1}^r.$$

If Assumption 1 is fulfilled, then V^d can be chosen such that the (r,r)-matrix $V^d U := [(v^i, u_j)]_{i,j=1}^r = I$, i.e., the basis $\{v^1, \ldots, v^r\}$ of $\operatorname{Ker}(L_d)$ can be chosen for dual basis to a basis $\{u_1, \ldots, u_r\}$ of $\operatorname{Ker}(L)$. If $V^d U = I$ then we can still replace U by UC and V^d by $C^{-1} V^d$ without destroying this dual pairing. Assumption (2°) says that no principal vectors w of L exist such that $Lw = u_i$; cf. Theorem **??**, 3°.
By Assumption 1(i) we have $\mathcal{E} = \operatorname{Range}(L) \oplus \mathcal{U}$, where $\dim \mathcal{U} = \dim \operatorname{Ker}(L_d) = \dim \operatorname{Ker}(L)$ is the dimension of the subspace $\mathcal{U}$. Because of Assumption 1(ii) we thus obtain again

$$\mathcal{E} = \operatorname{Range}(L) \oplus \operatorname{Ker}(L)$$

as in the previous parts of Chapter 5. This holds also if $L : \mathcal{E} \to \mathcal{F}$ and $\mathcal{E} \subseteq \mathcal{F}$. Let Q be again the projector

$$Q : \mathcal{E} \ni u \mapsto U(V^d u) \in \operatorname{Ker}(L)$$

then $\operatorname{Range}(Q) = \operatorname{Ker}(L)$ and $\operatorname{Range}(I - Q) = \mathit{Range}(L)$.

Lemma 1 (1°) *Adopt Assumption 1 and* $V^dU = I$.
(2°) *Let there exist exactly* r *different branches of characteristic pairs* $(\lambda_i(\varepsilon), u_i(\varepsilon))$ *such that* $(\lambda_i(0), u_i(0)) = (0, u_i)$, $i = 1 : r$ *which are continuously differentiable in some neighborhood of* $\varepsilon = 0$. *Then*

$$\langle v^i,\, Bu_j\rangle = \lambda_i'(0)\delta^i{}_j\,,\quad i,j = 1:r\,,$$

($\delta^i{}_j$ KRONECKER *symbol) hence* V^dBU *is a diagonal matrix.*

Proof.. Differentiation of (1) with respect to ε yields at the point $\varepsilon = 0$

$$Bu_k(0) + Lu_k'(0) = \lambda_k'(0)u_k(0) + \lambda_k(0)u_k'(0) = \lambda_k'(0)u_k(0).$$

Because $(v^i, u_k(0)) = (v^i, u_k) = \delta_{ik}$ and $(v^i, Lz) = (v^iL_d, z) = 0\ \forall\ z \in \mathcal{E}$, we obtain

$$(v^i, Bu_k) = \lambda_k'(0)(v^i, u_k) = \lambda_k'(0)\delta^i{}_k\,.\ \square$$

For the converse result, being somewhat more difficult, let $s \in \mathbb{N}$ arbitrary and let

$$\begin{array}{lcl} U(\varepsilon) & = & [u_1(\varepsilon),\ldots,u_r(\varepsilon)] \in \mathcal{E}^r\,,\\ W(\varepsilon) & = & [w_1(\varepsilon),\ldots,w_s(\varepsilon)] \in \mathcal{E}^s\,,\quad V^dW(\varepsilon) = 0\\ \Xi(\varepsilon) & \in & \mathbb{C}^r{}_s\,, \end{array}$$

be some preliminary unknown functions. Then the characteristic pairs $(\lambda(\varepsilon), U(\varepsilon)) \in \mathbb{C} \times \mathcal{E}^r$ are decomposed and scaled:

$$U(\varepsilon) = U\Xi(\varepsilon) + \varepsilon W(\varepsilon)\,,\quad \lambda(\varepsilon) = \varepsilon\omega(\varepsilon)\,,\quad V^dW(\varepsilon) = 0 \in \mathbb{R}^r{}_s\,,$$

such that $U(0) = U\Xi(0)$. We consider the matrix-eigenvalue problem

$$\boxed{L(\varepsilon)U(\varepsilon) = \lambda(\varepsilon)U(\varepsilon)} \tag{2}$$

in scaled form

$$\begin{array}{lcl} \Psi(\omega,\Xi,W,\varepsilon) & := & \varepsilon^{-1}[L(\varepsilon)U(\varepsilon) - \lambda(\varepsilon)U(\varepsilon)] \equiv LW + G(Z,\varepsilon) = 0\\ G(Z,\varepsilon) & = & r(\omega,\Xi) + r_1(Z,\varepsilon)\\ r(\omega,\Xi) & = & [B - \omega I]U\Xi \end{array}$$

where $Z = (\omega, \Xi, W) \in \mathcal{C} \times \mathcal{C}^{r\times s} \times \mathcal{E}^s$ and $r_1(Z, 0) = 0$. Let $Q = VV^d$ be the above introduced projector then the system

$$LW + (I - Q)G = 0\,,\quad V^dW = 0$$

has a unique solution $W \in [\text{Range}\, L]^s$ for every $G \in \mathcal{E}^s$. Let L^0 denote again the restriction of the FREDHOLM operator L to $\text{Range}\, L \subset \mathcal{E}$ then L^0 has a bounded inverse on $\text{Range}\, L$ by the Inverse Operator Theorem. By using L^0 , the system

$$(I - Q)\Psi(Z, \varepsilon) = LW + (I - Q)G(Z, \varepsilon) = 0\,,\quad V^dW = 0$$

is equivalent to

$$(I - Q)\Psi^0(Z, \varepsilon) := L^0W + (I - Q)G(Z, \varepsilon) = 0\,.$$

A solution of this system is solution of the original system $\Psi(Z, \varepsilon) = 0$ if $QG(Z, \varepsilon) = 0$, i.e., if $V^dG(Z, \varepsilon) = 0$. Therefore we consider as in the LJAPUNOV-SCHMIDT reduction the system

$$V^d\Psi(Z, \varepsilon) = 0\,,\quad (I - Q)\Psi(Z, \varepsilon) = 0\,.$$

Theorem 1 (1°) *Adopt Assumption 1.*
(2°) *Let* $V^d BU \in \mathbb{R}^{r\times r}$ *be diagonable.*
(3°) *Let* $(\omega^0, \Xi^0) \in \mathbb{C} \times \mathbb{C}^{r\times s}$ *be solution of the finite-dimensional matrix-eigenvalue problem*

$$[\omega I - V^d BU]\Xi = 0\,, \quad \Xi^H \Xi - I = 0 \tag{3}$$

where $s = \dim \operatorname{Ker}(\omega^0 I - V^d BU)$ *.*
Then the system

$$F(Z, \mu) := \begin{bmatrix} V^d \Psi(Z, \varepsilon) \\ \overline{\Xi^0}^T \Xi - I \\ (I - Q)\Psi^0(Z, \varepsilon) \end{bmatrix} = 0$$

has a unique solution $Z(\varepsilon) = (\omega(\varepsilon), \Xi(\varepsilon), W(\varepsilon))$ *for sufficiently small* $|\varepsilon|$ *and satisfies the initial condition* $(\omega(0), \Xi(0)) = (\omega^0, \Xi^0)$ *. The pair*

$$(\lambda(\varepsilon), U(\varepsilon)) := \big(\varepsilon\omega(\varepsilon), U\Xi(\varepsilon) + \varepsilon W(\varepsilon)\big)$$

is a nontrivial solution of the matrix-eigenvalue problem (2) such that $(\lambda(0), U(0)) = (0, U\Xi^0)$ *.*

Proof. The mapping

$$F : \mathcal{C} \times \mathcal{C}^{r\times s} \times [\text{Range}(L)]^s \times \mathcal{R} \to \mathcal{C} \times \mathcal{C}^{r\times s} \times [\text{Range}(L)]^s$$

has for $\varepsilon = 0$ the solution $Z(0) = (\omega^0, \Xi^0, W^0)$ where W^0 is the solution of $L^0 W + (I - Q)r(\omega, \Xi) = 0$. Because L^0 has a bounded inverse on its range, and because $(I - Q)r_{\omega,\Xi}(\omega^0, \Xi^0)$ is a finite-dimensional operator, the Frechet derivative

$$F_Z(Z(0), 0) = \begin{bmatrix} J(\omega^0, \Xi^0) & 0 \\ Q' r_{\omega,\Xi}(\omega^0, \Xi^0) & L^0 \end{bmatrix}$$

has a bounded inverse on the entire range of F if the Jacobi matrix of the system (3),

$$J(\omega^0, \Xi^0) = \begin{bmatrix} \Xi^0 & \omega^0 I - V^d BU \\ 0 & (\Xi^0)^H \end{bmatrix}, \tag{4}$$

is regular. Let $A = \omega^0 I - V^d BU$ then

$$\begin{aligned} \text{Range}(A) \cap \text{Ker}(A) &= \text{Range}(A) \cap \text{Range}(\Xi^0) = \{0\} \\ \text{Ker}(A) \cap \text{Ker}(A)^\perp &= \text{Ker}(A) \cap \text{Ker}[(\Xi^0)^H] = \{0\} \end{aligned} \tag{5}$$

because A is diagonable. Now, (5) is the assumption of the Bordering Lemma, Theorem 1.19, and thus an application of this lemma yields the desired regularity result for (4). The assertion of the theorem then follows by the Implicit Function Theorem and the solution is as smooth as F. □

The above announced smoothness result now follows directly from Theorem 1:

Corollary 1 *(i) Let Assumption 1 be fufilled.*
(ii) Let the matrix $V^d BU$ *be diagonable.*
Then r *branches* $(\lambda_i(x), u_i(x))$ *of the eigenvalue problem (1) with* $(\lambda_i(0), u_i(0)) = (0, u_i), i = 1 : r$*, are continuously differentiable in a neighborhood of* $\varepsilon = 0$*.*

Corollary 2 *(i) Let Assumption 1 be fulfilled.*
(ii) Let $\mathcal{E}$ be a Hilbert space and let $L(\varepsilon)$ be symmetric, $L(\varepsilon) = L_d(\varepsilon)$ for $\varepsilon \in \mathcal{I}$.
Then the matrix $V^d BU$ is hermitean hence diagonable.

Proof. In this case, B is self-adjoint, too, and $V = \overline{U}$ hence $V^d BU$ is a hermitean matrix. □

Let us now turn to the multi-parametric matrix eigenvalue problem

$$(L + \sum_{j=1}^{m} \varepsilon_j B_j)U(\varepsilon) = \lambda(\varepsilon)U(\varepsilon),\ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m). \tag{6}$$

If the problem is smooth and if $V^d U = I$ then all matrices $V^d B_j U, j = 1 : m$ must be diagonal. But, as U and V^d with $V^d U = I$ can still be replaced with UC and $C^{-1}V^d$ with any regular matrix C we have the following further corollary to Theorem 1:

Corollary 3 *Let Assumption 1 be fulfilled. Then r branches of characteristic pairs $(\lambda_i(\varepsilon), u_i(\varepsilon))$ of the eigenvalue problem (6) with $(\lambda_i(0), u_i(0)) = (0, u_i)$, $i = 1 : r$, exist and are continuously differentiable in a neighborhood of $\varepsilon = 0$ if and only if the matrices $V^d B_j U$, $j = 1 : m$, in (6) are commonly diagonable by a regular matrix C.*