Supplements 4 to Chapter V

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On the Newton-Polygon

Lemma 1 *Let the notations of section 5.3 hold and let Assumption 5.3 be fulfilled. If f is nonzero, nonlinear in (x, y), and $f(x, 0) \equiv 0$ then there exist rational numbers $r > 0$, $s > 0$, $t > 1$ such that the mappings Φ and ψ defined by*

$$\begin{aligned} \Phi(\xi, \zeta, w) &:= lim_{\varepsilon \to 0} \Phi(\xi, \zeta, w, \varepsilon), \\ Lw + \psi(\xi, \zeta) &:= lim_{\varepsilon \to 0} \Psi(\xi, \zeta, w, \varepsilon), \end{aligned}$$

exist and are polynomials in $(\xi, \zeta, w) \in \mathcal{R}^{m \times \nu} \times \mathcal{E}$.

Proof. We follow the idea of the Newton polygon, cf. e.g. [Chow], and consider the Taylor expansions

$$\begin{aligned} \varepsilon^r \Phi(\xi, \zeta, w, \varepsilon) &= \sum_{i=1}^{\varrho} \varepsilon^{a[i]} \varphi^{[i]}(\xi, \zeta, w), \\ \varepsilon^t \Psi(\xi, \zeta, w, \varepsilon) - \varepsilon^t Lw &= \sum_{j=1}^{\sigma} \varepsilon^{b[j]} \psi^{[j]}(\xi, \zeta, w) \end{aligned} \tag{1}$$

for sufficiently small $\varepsilon > 0$ and $(\xi, \zeta, w) \in \mathcal{R}^{m \times \nu} \times \mathcal{E}$. For simplicity we suppose that $\varrho, \sigma \in \mathcal{N} \cup \{\infty\}$ and that all multilinear mappings $\varphi^{[i]}, \psi^{[j]}$ do not disappear identically. Then

$$\begin{aligned} a[i] &= a_{i1}s + a_{i2} + a_{i3}t, \quad a_{ik} \in \mathcal{N}_0, \\ b[j] &= b_{j1}s + b_{j2} + b_{j3}t, \quad b_{jk} \in \mathcal{N}_0, \end{aligned}$$

and we also define the sets of integer tripels

$$\begin{aligned} A &:= \{a_i = (a_{i1}, a_{i2}, a_{i3}), i = 1, \dots, \varrho\}, \\ B &:= \{b_j = (b_{j1}, b_{j2}, b_{j3}), j = 1, \dots, \sigma\}, \end{aligned}$$

their union $C = A \cup B$, and the hyperplane

$$H = \{x \in \mathcal{R}^3,\ h^T x = p\}, \quad h = (h_1, h_2, h_3), \quad 0 \le p \in \mathcal{R}.$$

Let H be a support plane to C, i.e.,

$$\forall\, c \in C : \ h^T c \ge p, \quad \exists\, c \in C : \ h^T c = p.$$

Because C is a subset of integers of the closed first octant in $\mathcal{R}^3$, the number p and the components of h are rational numbers and H can always be chosen such that all these elements are nonnegative. Because of the nonlinearity of f the set C does not contain the elements $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ therefore there exists support planes H with positive h and p. We thus have only to show that we may choose $h_3 > h_2$ for such a support plane. Then the associated tripel

$$(r, s, t) = (p/h2, h_1/h_2, h_3/h_2)$$

has all desired properties. $C \cap H$ may consist of three points and more but this is not necessarily the case, in particular if C itself contains less than three points. As the normal vector h of H has positive components, H is moved away from C if any component of h is enlarged. E.g., if h_1 is enlarged then H is turned about the axis through $(0, p/h_2, 0)$ and $(0, 0, p/h_3)$. The contact to C is regained again by a movement in the direction of the normal vector h, i.e., by an enlargement of the right side p in the implicit representation $h^T x = p$ of H. Therefore the property $h_3 > h_2$ can be reached without destroying the support property of H. From this construction it is seen that Φ and ψ must be polynomials because all elements of C being not contained in H disappear in the limit $\varepsilon \to 0$.

It remains to show that ψ does not depend on w but this follows from the fact that in the case of existence of ψ

$$
\begin{aligned}
& lim_{\varepsilon\to 0}\nabla_w(\varepsilon^{-t}Q'M(\varepsilon^s\xi, \varepsilon V\zeta + \varepsilon^t w) \\
& = \nabla_w[lim_{\varepsilon\to 0}(\varepsilon^{-t}Q'M(\varepsilon^s\xi, \varepsilon V\zeta + \varepsilon^t w))] \\
& = M_y(0,0) = L.
\end{aligned}
$$

The proof shows that the scaled mappings Φ and ψ can be computed by a simple search program if the power series (1) are available. Under the above assumptions Φ and ψ cannot both be identically zero. If the defining support plane H does not contact the set A then $\Phi = 0$ and if it does not contact B then $\psi = 0$. A somewhat deeper study shows that Φ can only depend on w if the Taylor expansion (1) of $U^T f(\varepsilon^s\xi, \varepsilon V\zeta)$ has some gaps in comparison with the Taylor expansion of $f(\varepsilon^s\xi, \varepsilon y)$, i.e., if the powers $a[i]$ of both expansions do not agree.