Supplements 3 to Chapter V

E. Gekeler
11/10/06

To Section 5.3

(a) Recapitulation of Notations: We now consider problems with one or two parameters only hence

$$\boxed{\begin{array}{l} F : \mathcal{G} := \mathbb{R} \times \mathcal{E} \ni y = (\mu, x) \mapsto F(y) = F(\mu, x) \in \mathcal{F} \\ \text{resp.} \\ F : \mathcal{G} := \mathbb{R}^2 \times \mathcal{E} \ni y = (\mu, \omega, x) \mapsto F(y) = F(\mu, \omega, x) \in \mathcal{F} \end{array}} \tag{1}$$

at a point y_0 where $F(y_0) = 0$ and $y_0 = (\mu_0, x_0)$ resp. $y_0 = (\mu_0, \omega_0, x_0)$. We substitute a local solution $y(s)$ with $y(0) = y_0$ and derive w.r.t. the path parameter s then, at the point $s = 0$,

$$\begin{array}{l} F_\mu^0 \mu'(0) + F_x^0 x'(0) = 0 \\ F_\mu^0 \mu''(0) + F_x^0 x''(0) + F_{\mu\mu}^0 \mu'(0)^2 + 2F_{\mu x} \mu'(0) x'(0) + F_{xx}^0 \langle x'(0), x'(0) \rangle = 0\,, \end{array} \tag{2}$$

resp.

$$\begin{array}{l} F_\mu^0 \mu'(0) + F_\omega^0 \omega'(0) + F_x^0 x'(0) = 0 \\ F_\mu^0 \mu''(0) + F_\omega^0 \omega''(0) + F_x^0 x''(0) + F_{\mu\mu}^0 \mu'(0)^2 + F_{\omega\omega}^0 \omega'(0)^2 + 2F_{\mu x} \mu'(0) x'(0) \\ +2F_{\mu\omega}^0 \mu'(0) \omega'(0) + 2F_{\omega x}^0 \omega'(0) x'(0) + F_{xx}^0 \langle x'(0), x'(0) \rangle = 0\,. \end{array} \tag{3}$$

For instance, if $u \in \operatorname{Ker} F_x^0$ then $(0, u) \in \operatorname{Ker}[F_\mu^0,\, F_x^0]$. Further elements in $\operatorname{Ker} F_y^0$ exist in the single-parameter case only if $F_\mu^0 \in \operatorname{Range} F_x^0$. Recall also the notations

$$\boxed{\begin{array}{lclclcl} \operatorname{Ker} F_x^0 &=& \operatorname{span}\{u_1, \dots, u_\nu\}\,, & U &=& [u_1, \dots, u_\nu]\,, & U^d = [u^i]_{i=1}^\nu\,, \\ \operatorname{Ker}[F_x^0]_d &=& \operatorname{span}\{v^1, \dots, v^\nu\}\,, & V^d &=& [v^i]_{i=1}^\nu\,, & V = [v_1, \dots, v_\nu]\,, \\ \operatorname{Ker} F_y^0 &=& \operatorname{span}\{\widetilde{u}_1, \dots, \widetilde{u}_\mu\}\,, & \widetilde{U} &=& [\widetilde{u}_1, \dots, \widetilde{u}_\mu]\,, & \widetilde{U}^d = [\widetilde{u}^i]_{i=1}^\mu\,, \\ \operatorname{Ker}[F_y^0]_d &=& \operatorname{span}\{\widetilde{v}^1, \dots, \widetilde{v}^\kappa\}\,, & \widetilde{V}^d &=& [\widetilde{v}^i]_{i=1}^\kappa\,, & \widetilde{V} = [\widetilde{v}_1, \dots, \widetilde{v}_\kappa] \end{array}}\,. \tag{4}$$

Then by the e.g. for (1,1°)

$$\boxed{\dim \operatorname{Ker} F_x^0 \neq \dim \operatorname{Ker} F_y^0 \iff F_\mu^0 \in \operatorname{Range} F_x^0 \iff V^d F_\mu^0 = 0}\,.$$

$\dim \operatorname{Ker}[F_y^0]_d = 1$, one or two parameters:

$$\boxed{\begin{array}{llll} \text{Typ Ia:} & \dim \operatorname{Ker} F_x^0 = 1\,, & \dim \operatorname{Ker} F_y^0 = 2\,, & 0 = F_\mu^0 \ (\in \operatorname{Range} F_x^0) \\ \text{Typ Ib:} & \dim \operatorname{Ker} F_x^0 = 1\,, & \dim \operatorname{Ker} F_y^0 = 2\,, & 0 \neq F_\mu^0 \in \operatorname{Range} F_x^0 \\ \text{Typ Ic:} & \dim \operatorname{Ker} F_x^0 = 2\,, & \dim \operatorname{Ker} F_y^0 = 2\,, & 0 \neq F_\mu^0 \notin \operatorname{Range} F_x^0 \\ \text{Typ IIa:} & \dim \operatorname{Ker} F_x^0 = 2\,, & \dim \operatorname{Ker} F_y^0 = 3\,, & V^d F_\mu^0 \neq 0\,,\ V^d F_\omega^0 = 0 \\ \text{Typ IIb:} & \dim \operatorname{Ker} F_x^0 = 2\,, & \dim \operatorname{Ker} F_y^0 = 3\,, & V^d F_\mu^0 = 0\,,\ V^d F_\omega^0 \neq 0 \\ \text{Typ IIc:} & \dim \operatorname{Ker} F_x^0 = 3\,, & \dim \operatorname{Ker} F_y^0 = 3\,, & V^d F_\mu^0 \neq 0\,,\ V^d F_\omega^0 \neq 0 \end{array}}\,,$$

$\dim \mathrm{Ker}[F_y^0]_d = 2$, one parameter:

$$\boxed{\begin{array}{llll} \text{Typ IIIa:} & \dim \mathrm{Ker}\, F_x^0 = 2\,, & \dim \mathrm{Ker}\, F_y^0 = 3\,, & 0 = F_\mu^0\ (\in \mathrm{Range}\, F_x^0) \\ \text{Typ IIIb:} & \dim \mathrm{Ker}\, F_x^0 = 2\,, & \dim \mathrm{Ker}\, F_y^0 = 3\,, & 0 \neq F_\mu^0 \in \mathrm{Range}\, F_x^0 \\ \text{Typ IIIc:} & \dim \mathrm{Ker}\, F_x^0 = 3\,, & \dim \mathrm{Ker}\, F_y^0 = 3\,, & 0 \neq F_\mu^0 \notin \mathrm{Range}\, F_x^0 \end{array}}\ .$$

Necessary branching conditions shall now be derived for these types in the same way as in **(a)**. To this end the subspace $\mathrm{Ker}\, F_y^0 = \mathrm{span}\{\widetilde{v}^1, \dots, \widetilde{v}^\kappa\}$ resp. the matrix $\widetilde{V}^d = [\widetilde{v}^1, \dots, \widetilde{v}^\kappa]$ has to be found explicitely. The same matrix applies also in the the projector $Q = \widetilde{V}\widetilde{V}^d$ where $\mathrm{Ker}\, Q = \mathrm{Range}\, F_y^0$, being needed in the branching equation .

(b) Computation of Quadratic Forms for Type I

(b1, Type Ia/b) $v^1 F_\mu^0 = 0$, hence $v^1 F_y^0 = v^1 [F_\mu^0,\, F_x^0] = (0,0) \in \mathbb{R} \times \mathcal{E}_d$. Let w be the unique solution of $F_\mu^0 + F_x^0 w = 0$ where $u^1\, w = 0$ then

$$\widetilde{U} = \begin{bmatrix} 0 & 1 \\ u_1 & w \end{bmatrix}, \quad \widetilde{V}^d = v^1 =: \widetilde{v}^d\,. \tag{5}$$

Every tangent $x'(0)$ in type Ib has the representation $x'(0) = \alpha u_1 + \beta w$ where $\beta = \mu'(0)$. Insertion of $(\beta, x'(0))$ in (2) yields after multiplication by the *vector* $\widetilde{v}^d$ of (5)

$$\widetilde{v}^d F_{\mu\mu}^0 \beta^2 + 2\widetilde{v}^d F_{\mu x}^0 \beta(\alpha u_1 + \beta w) + \widetilde{v}^d F_{xx}^0 \langle \alpha u_1 + \beta w, \alpha u_1 + \beta w\rangle = 0\,.$$

or $a^T Q_3(u_1, w, \widetilde{v}^d) a = 0\,,\ a^T = [\alpha,\, \beta]$ where

$$Q_3(u_1, w, \widetilde{v}^d) = \begin{bmatrix} \widetilde{v}^d F_{xx}^0 \langle u_1, u_1\rangle & \widetilde{v}^d \big[F_{\mu x}^0 u_1 + F_{xx}^0 \langle u_1, w\rangle\big] \\ \widetilde{v}^d \big[F_{\mu x}^0 u_1 + F_{xx}^0 \langle u_1, w\rangle\big] & \widetilde{v}^d \big[F_{\mu\mu}^0 + 2F_{\mu x}^0 w + F_{xx}^0 \langle w, w\rangle\big] \end{bmatrix}. \tag{6}$$

This quadratic form has two different real solutions in case
$\det(Q_3(u_1, w, \widetilde{v}^d)) < 0$; in type Ia $w = 0$.

(b2, Type Ic) $V^d F_\mu^0 \neq 0 \in \mathbb{R}^2$ hence there exists a vector $0 \neq a \in \mathbb{R}_2$ such that $a[V^d F_\mu^0] = 0$, and then $aV^d F_y^0 = aV^d [F_\mu^0,\, F_x^0] = (0,0) \in \mathbb{R} \times \mathcal{E}_d$; further linearly independent left-vectors v satisfying $vF_y^0 = 0$ are not available. By (2),1° now $\mu'(0) = 0$ after multiplication by V^d:

$$\mu'(0) = 0\,,\ \ \widetilde{U} = \begin{bmatrix} 0 & 0 \\ u_1 & u_2 \end{bmatrix}, \quad \widetilde{V}^d = aV^d =: \widetilde{v}^d\,. \tag{7}$$

Every tangent has the representation $x'(0) = \alpha u_1 + \beta u_2$. By (2) after multiplication by the vector $\widetilde{v}^d = aV^d$ of (7)

$$\begin{aligned} &\widetilde{v}^d F_{xx}^0 \langle \alpha u_1 + \beta u_2, \alpha u_1 + \beta u_2\rangle \\ &= \widetilde{v}^d F_{xx}^0 \langle u_1, u_1\rangle \alpha^2 + 2\widetilde{v}^d F_{xx}^0 \langle u_1, u_2\rangle \alpha\beta + \widetilde{v}^d F_{xx}^0 \langle u_2, u_2\rangle \beta^2 = 0 \end{aligned}$$

or $a^T Q_4(u_1, u_2, \widetilde{v}^d) a = 0\,,\ a^T = [\alpha,\, \beta]$, where

$$Q_4(u_1, u_2, \widetilde{v}^d) = [\widetilde{v}^d F_{xx}^0 \langle u_i, u_k\rangle]_{i,k=1}^2\,. \tag{8}$$

Corollary 1 *(Necessary Conditions)*

(1°) *Let* (μ_0, x_0) *be a branching point of type Ia then* $\det(Q_3(u_1, 0, \widetilde{v}^d)) < 0$.

(2°) *Let* (μ_0, x_0) *be a branching point of type Ib then*
$\det(Q_3(u_1, w, \widetilde{v}^d) < 0$.

(3°) *Let* (μ_0, x_0) *be a branching point of type Ic then*
$\det(Q_4(u_1, u_2, \widetilde{v}^d)) < 0$, *and both tangents lie in the hyperplane* $\mu = \mu_0$.

(c) Computation of Quadratic Forms for Type II and Type III

(b4, Type IIa) $V^d = [v^1,\, v^2]\,, V^d F_\mu^0 \neq 0\,,\, V^d F_\omega^0 = 0$. There exists a vector $0 \neq a \in \mathbb{R}_2$ with $aV^d F_\mu^0 = 0$, then $aV^d[F_\mu^0,\, F_\omega^0, F_x^0] = 0$; further linearly independent vectors with this property do not exist. Let w be the unique solution of $F_\omega^0 + F_x^0 w = 0$ mit $U^d w = 0\,$, then

$$\mu'(0) = 0\,,\;\; \widetilde{U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ u_1 & u_2 & w \end{bmatrix}\,,\;\; \widetilde{V}^d = aV^d =: \widetilde{v}^d\,. \tag{9}$$

Every tangent has the unique representation $x'(0) = \alpha u_1 + \beta u_2 + \gamma w$ where $\gamma = \omega'(0)$. Inserting of $(0,\, \gamma,\, x'(0))$ in (3) yields after multiplication by $\widetilde{v}^d$ of (9) the quadratic form

$$a^T Q_5(u_1, u_2, w, \widetilde{v}^d)a = 0\,,\;\; a^T = [\alpha, \beta, \gamma]\,,$$

$$Q_5(u_1, u_2, w, \widetilde{v}^d) =$$

$$\begin{bmatrix} \widetilde{v}^d F_{xx}^0\langle u_1, u_1\rangle & \widetilde{v}^d F_{xx}^0\langle u_1, u_2\rangle & \widetilde{v}^d\big[F_{xx}^0\langle u_1, w\rangle + F_{\omega x}^0 u_1\big] \\ \widetilde{v}^d F_{xx}^0\langle u_2, u_1\rangle & \widetilde{v}^d F_{xx}^0\langle u_2, u_2\rangle & \widetilde{v}^d[F_{xx}^0\langle u_2, w\rangle + F_{\omega x}^0 u_2] \\ \widetilde{v}^d\big[F_{\omega x}^0 u_1 + F_{xx}^0\langle w, u_1\rangle\big] & \widetilde{v}^d\big[F_{\omega x}^0 u_2 + F_{xx}^0\langle w, u_2\rangle\big] & \widetilde{v}^d\big[F_{\omega\omega}^0 + F_{xx}^0\langle w, w\rangle\big] \end{bmatrix}\,, \tag{10}$$

which must describe a cone in $\mathbb{R}^3$.

(b5, Type IIb) $V^d = [v^1,\, v^2]\,,\;\; V^d F_\mu^0 = 0\,,\;\; V^d F_\omega^0 \neq 0\,$. Let w be the unique solution of $F_\mu^0 + F_x^0 w = 0$ with $U^d w = 0\,$, then

$$\omega'(0) = 0\,,\;\; \widetilde{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ u_1 & u_2 & w \end{bmatrix}\,,\;\; \widetilde{V}^d = aV^d =: \widetilde{v}^d\,. \tag{11}$$

Accordingly, in the matrix $Q_5(u_1, u_2, \widetilde{v}^d)$ the partial derivatives w.r.t. ω are to be replaced by the partial derivatives w.r.t. μ.

(b6, Typ IIc) $V^d = [v^1,\, v^2,\, v^3]\,,\;\; V^d F_\mu^0 \neq 0\,,\;\; V^d F_\omega^0 \neq 0\,$. There exists a vector $0 \neq a \in \mathbb{R}^3$ such that $aV^d F_\mu^0 = 0$ and $aV^d F_\omega^0 = 0$, then $aV^d[F_\mu^0,\, F_\omega^0,\, F_x^0] = 0$; further linearly independent vectors with this property do not exist:

$$\mu'(0) = \omega'(0) = 0\,,\;\; \widetilde{U} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_1 & u_2 & u_3 \end{bmatrix}\,,\;\; \widetilde{V}^d = aV^d =: \widetilde{v}^d\,. \tag{12}$$

Every tangent has again the form $x'(0) = \alpha u_1 + \beta u_2 + \gamma u_3\,$. Inserting in (3) yields after multiplication by $\widetilde{v}^d$ of (12) the quadratic form

$$a^T Q_6(u_1, u_2, u_3, \widetilde{v}^d)a = 0\,,\;\; a^T = [\alpha,\, \beta,\, \gamma]\,,\;\; Q_6(u_1, u_2, u_3, \widetilde{v}^d) = [\widetilde{v}^d F_{xx}^0\langle u_i, u_k\rangle]_{i,k=1}^3\,, \tag{13}$$

which must be describe a cone again.

(b7, Typ IIIa/b) $V^d F_\mu^0 = 0$ with $V^d = [v^1,\, v^2]^T$, hence $V^d F_y^0 = V^d[F_\mu^0,\, F_x^0] = (0, 0)\,$. Besides v^1 and v^2 there do not exist further vectors v such that $vF_y^0 = 0$. Let w be the unique solution of $F_\mu^0 + F_x^0 w = 0$ such that $U^d w = 0\,$, then

$$\widetilde{U} = \begin{bmatrix} 0 & 0 & 1 \\ u_1 & u_2 & w \end{bmatrix}\,,\;\; \widetilde{V}^d = V^d\,. \tag{14}$$

For type Typ IIIb every tangent has the form $x'(0) = \alpha u_1 + \beta u_2 + \gamma w$ where $\gamma = \mu'(0)$. Inserting of $(\gamma, x'(0))$ in (2) yields after multiplication by $\widetilde{V}^d$ of (14) the both quadratic forms

$$a^T Q_5(v^i, w)a = 0\,,\ a^T = [\alpha, \beta, \gamma]\,,\ i = 1, 2\,, \tag{15}$$

which must satisfy the cone condition of **(a)**.
In Type IIIa we have $w = 0$, therefore $x'(0)$ has the representation $x'(0) = \alpha u_1 + \beta u_2$. Inserting of $(\gamma, x'(0))$ in (2) yields after multiplication by $V^d = [v^1,\, v^2]$ the both quadratic forms

$$a^T Q_5(v^i, 0)a = 0\,,\ a^T = [\alpha, \beta, \gamma]\,,\ i = 1, 2\,. \tag{16}$$

(b8, Type IIIc) $V^d F^0_\mu \neq 0$ where $V^d = [v^1,\, v^2,\, v^3]^T$, therefore there exist exactly two linearly independent vectors a, $b \in \mathbb{R}_3$ such that $aV^d F^0_\mu = bV^d F^0_\mu = 0$:

$$\mu'(0) = 0\,,\quad \widetilde{U} = \begin{bmatrix} 0 & 0 & 0 \\ u_1 & u_2 & u_3 \end{bmatrix}\,,\quad \widetilde{V}^d = \begin{bmatrix} aV^d \\ bV^d \end{bmatrix}\,. \tag{17}$$

Every tangent has now the representation $x'(0) = \alpha u_1 + \beta u_2 + \gamma u_3$. Inserting of $(0, x'(0))$ in (2) yields after multiplication by $\widetilde{V}^d$ of (17) the both quadratic forms

$$a^T Q_6(\widetilde{v}^i)a = 0\,,\ i = 1, 2\,, \tag{18}$$

cf. (13), which must satisfy the cone condition. The tangents then lie in the hyperplane $\mu = \mu_0$.

(d) Computation of Branching Points of Type I
We prove at first a result for the accompanying system (5.21), namely

$$\Phi_2(z) := \Phi_2(\mu, x, v) := \begin{bmatrix} F(\mu, x) \\ v\, F_x(\mu, x)) \\ v\, F_\mu(\mu, x)) - 1 \end{bmatrix} = 0\,,\ z = (\mu, x, v)\,,\ v \in \mathcal{F}_d\,, \tag{19}$$

where F^0_x is a Fredhom operator with index one by exception.

Lemma 1 *Let $F^0_x \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ be a* Fredhom *operator with index one and*

$$\begin{aligned} \dim \mathrm{Ker}\, F^0_x &= 2\,,\quad \mathrm{Ker}\, F^0_x = \mathrm{span}\{u_1, u_2\}\,, \\ \dim \mathrm{Ker}[F^0_x]_d &= 1\,,\quad \mathrm{Ker}[F^0_x]_d = \mathrm{span}\{v^1\}\,, \end{aligned}$$

and let $\Phi_2(z_0) = 0$ for $z_0 = (\mu_0, x_0, v^1)$. Then $\mathrm{grad}\, \Phi_2(z_0)$ *has a bounded inverse if and only if the matrix*

$$Q_4(u_1, u_2, v^1) = [v^1 F^0_{xx}\langle u_i, u_k\rangle]^2_{i,k=1} \tag{20}$$

is regular.

Note that $v^1 F^0_\mu \neq 0$ by the assumption $\Phi(z_0) = 0$.
Proof. The proof is carried out in much the same way as in Lemma 1 of `SUPPLEMENT\chap05b`. Let again

$$\begin{aligned} z &= (\sigma, z_1, z^2) \in \mathbb{R} \times \mathcal{E} \times \mathcal{F}_d\,, \\ g &= (g_1, g^2, \tau) \in \mathrm{Range}\, F^0_x \times \mathrm{Range}[F^0_x]_d \times \mathbb{R}\,. \end{aligned}$$

Then we have to show that the linear system $\operatorname{grad}\Phi_2(z^0)z = g\,,\ z^0 = (\sigma, z_1, z^2)\,,$ has a unique solution z for every g. In detail

$$\begin{array}{lcl} \operatorname{grad}\Phi_2(z^0)z = F^0_\mu\sigma + F^0_x z_1 & = & g_1 \\ v^1\big[F^0_{\mu x}\sigma + F^0_{xx}z_1\big] + z^2F^0_x & = & g^2 \\ v^1\big[F^0_{\mu\mu}\sigma + F^0_{\mu x}z_1\big] + z^2F^0_\mu & = & \tau\,. \end{array} \tag{21}$$

We choose the decomposition

$$\begin{array}{lcl} z_1 & = & \alpha u_1 + \beta u_2 + w\,,\quad U^d w = 0\,,\qquad U^d U = I_2\,, \\ z_2 & = & \gamma v^1 + q\,,\qquad\qquad \langle q,\, v_1\rangle = 0\,,\quad v^1 v_1 = 1\,; \end{array}$$

then we have to show that $\sigma,\ \alpha,\ \beta,\ \gamma, w, q$ are uniquely determined. Inserting into (21) yields

$$\begin{array}{lcl} F^0_\mu\sigma + F^0_x(\alpha u_1 + \beta u_2 + w) & = & g_1 \\ v^1\big[F^0_{\mu x}\sigma + F^0_{xx}(\alpha u_1 + \beta u_2 + w)\big] + z^2F^0_x & = & g^2 \\ v^1\big[F^0_{\mu\mu}\sigma + F^0_{\mu x}(\alpha u_1 + \beta u_2 + w\big] + z^2F^0_x & = & \tau\,. \end{array} \tag{22}$$

(1°) We have $v^1F^0_x = 0$ and $v^1F^0_\mu \neq 0$ because $F^0_\mu \notin \operatorname{Range} F^0_x$, and $v^1g_1 = 0$ because $g_1 \in \operatorname{Range} F^0_x$; therefore the first equation of (22) shows that $\sigma = v^1g_1/v^1F^0_\mu = 0$.
(2°) The component w is therefore the unique solution of

$$F^0_x w = g_1\,,\ U^d w = 0\,.$$

(3°) Inserting $\sigma = 0$ and the expression for z^2 in (22,2°) yields

$$(\beta v^1 + q)F^0_x = g^2 - v^1F^0_{xx}(\alpha u_1 + \beta u_2 + w) =: b \in \operatorname{Range}[F^0_x]_d\,. \tag{23}$$

By the Range Theorem $bu_1 = bu_2 = 0$ and $g^2u_1 = g^2u_2 = 0$. Application of (23) to u_1 and u_2 yields the system

$$\begin{array}{lcl} v^1F^0_{xx}\langle u_1, u_1\rangle\alpha + v^1F^0_{xx}\langle u_2, u_1\rangle\beta & = & v^1F^0_{xx}\langle w, u_1\rangle \\ v^1F^0_{xx}\langle u_1, u_2\rangle\alpha + v^1F^0_{xx}\langle u_2, u_2\rangle\beta & = & v^1F^0_{xx}\langle w, u_2\rangle \end{array}$$

which has a unique solution $\alpha,\ \beta$ iff the matrix Q_4 is regular.
(4°) Thus z_1 is determined uniquely. Because $v^1F^0_x = 0$ then q is the unique solution of (23) such that $\langle q, v_1\rangle = 0$. Finally, γ is uniquely determined from

$$(\gamma v^1 + q)F^0_\mu = \tau - v^1F^0_{\mu x}z_1\,.$$

□

By a proposition of [MooreA], branching points are computed as *regular* points of an *augmented* accompanying system in a similar way as in the computation of turning points. We consider the perturbated system

$$\Phi_3(\lambda, (\mu, x)) := \Phi_2(\mu, x) + \lambda r = 0\,, \tag{24}$$

where λ now plays the role of the former parameter μ and (μ, x) the role of the former x. We have supposed in Lemma 1 that $v^1F^0_\mu \neq 0$. Now μ is replaced by λ and

$$\frac{\partial}{\partial\lambda}\Phi_3(0, (\mu^0, x^0)) = r$$

To apply Lemma 1 to (24) we therefore require that the fixed and *specified* r is chosen such that $v^1 r \neq 0$ which is entirely natural for the following system. The accompanying system then reads with $v \in \mathcal{F}_d$:

$$\boxed{\begin{aligned}&\Phi_3(z) := \Phi_3(\lambda, (\mu, x), v) \\ &= \begin{bmatrix} F(\mu, x) + \lambda r \\ v\,[F_\mu(\mu, x),\ F_x(\mu, x)] \\ v\,r - 1 \end{bmatrix} = \begin{bmatrix} F(\mu, x) + \lambda r \\ v\,F_x(\mu, x) \\ v\,F_\mu(\mu, x) \\ v\,r - 1 \end{bmatrix} = 0\end{aligned}} \,. \tag{25}$$

Lemma 2 *Let* (μ_0, x_0) *be a branching point of type Ia/b and let* $z_0 = (0, \mu_0, x_0, v^1)$, *cf. (5), then* $\Phi_3(z_0) = 0$. *Further, let* w *denote the unique solution of* $F^0_\mu + F^0_x w = 0$ *where* $v^1\,w = 0$ *then*

$$\operatorname{grad}\Phi_3(z_0) : \mathbb{R}^2 \times \mathcal{E} \times \mathcal{F}_d \to \mathcal{F} \times \mathcal{F}_d \times \mathbb{R}^2$$

is regular if and only if the matrix $Q_3(u_1, w, v^1)$ *in (6) is regular.*

Proof. Let $F^0_\mu + F^0_x w = 0$ then

$$\operatorname{Ker}([F^0_\mu, F^0_x]) = \operatorname{span}\{(0, u_1), (1, w)\} =: \operatorname{span}\{\widetilde{u}_1, \widetilde{u}_2\}. \tag{26}$$

Then Lemma 1 w.r.t. the augmented system says that the matrix $Q_4(\widetilde{u}_1, \widetilde{u}_2, v^1)$ in (8) must be regular. Inserting the values for $\widetilde{u}_1$, $\widetilde{u}_2$ shows that $Q_4(\widetilde{u}_1, \widetilde{u}_2, v^1) = Q_3(u_1, w, v^1)$ where Q_3 is the matrix in (6)

Lemma 3 *Let* (μ_0, x_0) *be a branching point of type type Ic and let* $z_0 = (0, \mu_0, x_0, \widetilde{v}^d)$, *cf. (7). Then* $\Phi_3(z_0) = 0$ *and*

$$\operatorname{grad}\Phi_3(z_0) : \mathbb{R}^2 \times \mathcal{E} \times \mathcal{F}_d \to \mathcal{F} \times \mathcal{F}_d \times \mathbb{R}^2$$

is regular if and only if the matrix $Q_4(u_1, u_2, \widetilde{v}^d)$ *in (8) is regular.*

Hint to the proof. In the present case

$$\operatorname{Ker}([F^0_\mu, F^0_x]) = \operatorname{span}\{(0, u_1), (0, u_2)\}\,.$$

By consequence, the matrix Q_4 in (8) relative to the augmented system has now the form Q_4 of (8) relative to the original system. □

(d) Computation of Branching Points of Type II
We prove at first an auxiliary result for the system

$$\boxed{\Phi_4(z) := \Phi_4(\mu, x, v) := \begin{bmatrix} F(\mu, x) \\ v\,F_x(\mu, x)) \\ v\,F_\mu(\mu, x)) - 1 \end{bmatrix} = 0\,,\ z = (\mu, x, v)\,,\ v \in \mathcal{F}_d} \tag{27}$$

where, by exception, F_x^0 is a Fredhom operator with *index two*:

Lemma 4 *Let $F_x^0 \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ be a* Fredhom *operator with index two and*

$$\begin{aligned} \dim\operatorname{Ker} F_x^0 &= 3\,, \quad \operatorname{Ker} F_x^0 &&= \operatorname{span}\{u_1, u_2, u_3\} \\ \dim\operatorname{Ker}[F_x^0]_d &= 1\,, \quad \operatorname{Ker}[F_x^0]_d &&= \operatorname{span}\{v^1\}\,. \end{aligned}$$

and let $\Phi_4(z^0) = 0$ for $z_0 = (\mu_0, x_0, v^1)$. Then $\operatorname{grad}\Phi_4(z^0)$ *has a bounded inverse iff the matrix (13),*

$$Q_7(u_1, u_2, u_3, v^1) = \begin{bmatrix} v^1F_{xx}^0(u_1, u_1) & v^1F_{xx}^0(u_1, u_2) & v^1F_{xx}^0(u_1, u_3) \\ v^1F_{xx}^0(u_2, u_1) & v^1F_{xx}^0(u_2, u_2) & v^1F_{xx}^0(u_2, u_3) \\ v^1F_{xx}^0(u_3, u_1) & v^1F_{xx}^0(u_3, u_2) & v^1F_{xx}^0(u_3, u_3) \end{bmatrix}, \tag{28}$$

is regular.

Note that $v^1F_\mu^0 = 1$ by assumption that $\Phi_4(z_0) = 0$.
Proof. The proof is carried out essentially in the same way as Lemma 1. Let again

$$\begin{aligned} z &= (\sigma, z_1, z^2) \in \mathbb{R} \times \mathcal{E} \times \mathcal{F}_d\,, \\ g &= (g_1, g^2, \tau_1) \in \operatorname{Range} F_x^0 \times \operatorname{Range}[F_x^0]_d \times \mathbb{R}\,. \end{aligned}$$

Then we have to show that the linear system System $\operatorname{grad}\Phi_4(z^0)z = g$, $z^0 = (\sigma, z_1, z^2)$, has a unique solution z for every right side g. In detail

$$\begin{aligned} F_\mu^0\sigma + F_x^0 z_1 &= g_1 \\ v^1\big[F_{x\mu}^0\sigma + F_{xx}^0 z_1\big] + z^2F_x^0 &= g^2 \\ v^1\big[F_{\mu\mu}^0\sigma + F_{\mu x}^0 z_1\big] + z^2F_\mu^0 &= \tau_1\,. \end{aligned} \tag{29}$$

We choose the decomposition

$$\begin{aligned} z_1 &= Ua + w\,, \quad U^d w = 0\,, \qquad U^dU = I_3\,,\ U = [u_1, u_2, u_3]\,, \\ z_2 &= \gamma v^1 + q\,, \quad \langle q,\, v_1\rangle = 0\,, \quad v^1v_1 = 1\,; \end{aligned}$$

then it is to show that σ, γ, $a \in \mathbb{R}^3$ as well as w and q are determined uniquely. Inserting into (29) yields

$$\begin{aligned} F_\mu^0\sigma + F_x^0(Ua + w) &= g_1 \\ v^1\big[F_{x\mu}^0\sigma + F_{xx}^0(Ua + w)\big] + z^2F_x^0 &= g^2 \\ v^1\big[F_{\mu\mu}^0\sigma + F_{\mu x}^0(Ua + w)\big] + z^2F_\mu^0 &= \tau_1\,. \end{aligned} \tag{30}$$

(1°) The first equation supplies $\sigma = 0$ after multiplication by v^1 since $v^1F_x^0 = 0$ and $v^1g_1 = 0$.
(2°) Inserting of $\sigma = 0$ into the first equation yields w with $U^dw = 0$ uniquely because $F_x^0U = 0$.
(3°) Inserting $\sigma = 0$ and z_1 in (30,2°) yields

$$(\gamma v^1 + q)F_x^0 = g^2 - v^1F_{xx}^0(Ua + w) \in \operatorname{Range}[F_x^0]_d\,. \tag{31}$$

But $v^1 F^0 x = 0$, $F_x^0 u_i = 0$ and by the Range Theorem $g^2 u_i = 0$ Application of (31) to u_i, $i = 1:3$, successively shows that $a \in \mathbb{R}^3$ exists uniquely if the matrix Q_7 is regular.
(4°) Then q is the unique solution of (31) with $\langle q,\, v_1\rangle = 0$.
(5°) γ is uniquely determined in linear dependence of σ by

$$(\gamma v^1 + q)F_\mu^0 = \tau_1 - v^1 F_{\mu x}^0 z_1 \,. \square$$

Type IIa $\dim \operatorname{Ker} F_x^0 = 2$, $\widetilde{v}^d F_\mu^0 = 0$, $\widetilde{v}^d F_\omega^0 = 0$, cf. (9). We consider the perturbed system

$$\begin{aligned}
&\Phi_4(z) := \Phi_4(\lambda, (\mu,\omega,x), v)\\
&= \begin{bmatrix} F(\mu,\omega,x) + \lambda r \\ v\,[F_\omega(\mu,\omega,x),\; F_x(\mu,\omega,x)] \\ v\,F_\mu(\mu,\omega,x) \\ v\,r - 1 \end{bmatrix} \simeq \begin{bmatrix} F(\mu,\omega,x) + \lambda r \\ v\,F_x(\mu,\omega,x) \\ v\,F_\omega(\mu,\omega,x) \\ v\,F_\mu(\mu,\omega,x) \\ v\,r - 1 \end{bmatrix} = 0
\end{aligned} \tag{32}$$

where λ play the role of the parameters μ in Lemma 4 and $\widetilde{x} := (\mu,\omega,x)$ plays the role of the former element x. The vector r must be specified such that $\widetilde{v}^d r \neq 0$.

Lemma 5 *Let (μ_0,ω_0,x_0) be a branching point of type IIa and let $z_0 = (0,\mu_0,\omega_0,x_0,\widetilde{v}^d)$, cf. (9), then $\Phi_4(z^0) = 0$. Let moreover w be the unique solution of $F_\omega^0 + F_x^0 w = 0$ with $Uw = 0$, then*

$$\operatorname{grad}\Phi_4(z_0) : \mathbb{R}^3 \times \mathcal{E} \times \mathcal{F}_d \to \mathcal{F} \times \mathcal{F}_d \times \mathbb{R}^3$$

has a bounded inverse iff the matrix $Q_5(u_1,u_2,w,\widetilde{v}^d)$ in (10) is regular and if $\widetilde{v}^d r \neq 0$.

Proof. For the mentioned w we have by (9)

$$F_y^0 = [F_\mu^0, F_\omega^0, F_x^0]\,,\quad \operatorname{Ker}(F_y^0) = \operatorname{span}\{(0,0,u_1),(0,0,u_2),(0,1,w)\} =: \operatorname{span}\{\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3\} \tag{33}$$

Let $\widetilde{Q}_7(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{v}^d)$ be the matrix in (28) with partial derivatives w.r.t. $\widetilde{x} = (\mu,\omega,x)$ instead of x. Then Lemma 4 w.r.t. the augmented system (32) says that this matrix must be regular. Inserting the values of (33) we find that

$$\widetilde{Q}_7(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{v}^d) = Q_5(u_1,u_2,w,\widetilde{v}^d)\,. \square$$

For **Type IIb** we have to permute μ and ω and $\widetilde{v}^d$ is to be chosen as in (11).
Type IIc $\dim \ker F_x^0 = 3$, $\widetilde{v}^d F_\mu^0 = 0$, $\widetilde{v}^d F_\omega^0 = 0$, cf. (11). We consider again the perturbed system (32).

Lemma 6 *Let (μ_0,ω_0,x_0) be a branching point of type IIc and let $z_0 = (0,\mu_0,\omega_0,x_0,\widetilde{v}^d)$, cf. (12), then $\Phi_4(z^0) = 0$ and*

$$\operatorname{grad}\Phi_4(z_0) : \mathbb{R}^3 \times \mathcal{E} \times \mathcal{F}_d \to \mathcal{F} \times \mathcal{F}_d \times \mathbb{R}^3$$

has a bounded inverse iff the matrix $Q_6(u_1,u_2,u_3,\widetilde{v}^d)$ in (13) or (28) is regular and if $\widetilde{v}^d r \neq 0$ and a system-inherent constant is non-zero.

Proof. By (12) we have

$$F_y^0 = [F_\mu^0, F_\omega^0, F_x^0]\,, \quad \mathrm{Ker}(F_y^0) = \mathrm{span}\{(0,0,u_1),(0,0,u_2),(0,0,u_3)\} =: \mathrm{span}\{\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3\} \qquad (34)$$

Lemma 4 w.r.t. the augmented system (32) says again that the matrix $\widetilde{Q}_7(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{v}^d)$ in (28) must be regular. Inserting the values of (34) we find that

$$\widetilde{Q}_7(\widetilde{u}_1,\widetilde{u}_2,\widetilde{u}_3,\widetilde{v}^d) = Q_6(u_1,u_2,u_3,\widetilde{v}^d)\,. \; \square$$