Supplements 2 to Chapter V

E. Gekeler
11/10/06

To Section 5.2

(c) To **Compute Turning Points**, [Keener73] and [MooreB] have proposed *augmented accompanying systems*. The first one reads:

$$\boxed{\Phi_1(z) := \Phi_1(\mu, x, u) := \begin{bmatrix} F(\mu, x) \\ F_x(\mu, x)u \\ a\,u - 1 \end{bmatrix} = 0\,, \quad z = (\mu, x, u)\,,\ a \in \mathcal{E}_d \text{ fixed}}\,. \tag{1}$$

Lemma 1 *Let (μ_0, x_0) be a turning point and $a\,u_1 = 1$. Then* $\Phi_1(z_0) = \Phi_1(\mu_0, x_0, u_1) = 0$ *and*

$$\operatorname{grad}\Phi_1(z_0) : \mathbb{R} \times \mathcal{E} \times \mathcal{E} \to \mathcal{F} \times \mathcal{F} \times \mathbb{R}$$

has a bounded inverse if and only if

$$v^1 {F^0}_{xx}\langle u_1, u_1\rangle \neq 0\,. \tag{2}$$

Proof. Cf. [MooreB]. We show that $[\Phi_1]_z(z_0)$ is bijective iff (2) holds. Then the assertion follows from the Inverse Operator Theorem because $[\Phi_1]_z(z^0)$ is bounded and linear. Let

$$z = (\sigma, z_1, z_2) \in \mathbb{R} \times \mathcal{E} \times \mathcal{E}, \quad g = (g_1, g_2, \tau) \in \mathcal{F} \times \mathcal{F} \times \mathbb{R}$$

and consider the linear system

$$[\Phi_1]_z(z_0)z = g\,, \quad z_0 = (\mu_0, x_0, u), .$$

An expansion of this equation yields

$$F^0_\mu\sigma + F^0_x z_1 = g_1, \tag{3}$$

$$F^0_{x\mu}u_1\sigma + F^0_{xx}\langle u_1, z_1\rangle + F^0_x z_2 = g_2, \tag{4}$$

$$az_2 = \tau. \tag{5}$$

In order to show that z_1 and z_2 are determined uniquely, we can choose an arbitrary but fixed decomposition of the space $\mathcal{E}$. We use a decomposition defined by a and write

$$\begin{aligned} z_1 &= \alpha u_1 + w, \quad a\,w = 0, \\ z_2 &= \beta u_1 + q, \quad a\,q = 0, \end{aligned} \tag{6}$$

and show that σ, α, β, w, and q are determined uniquely. From (3) we obtain

$$\sigma = v^1 g_1 / v^1 F^0_\mu$$

because $v^1 F^0_x = 0$, and w is the unique solution of

$$F^0_x w = g_1 - F^0_\mu\sigma, \quad a\,w = 0.$$

Insertion of σ and z_1 into (4) yields

$$F^0_x(\beta u_1 + q) = g_2 - F^0_{x\mu}u_1\sigma - F^0_{xx}\langle u_1, \alpha u_1 + w\rangle =: b. \tag{7}$$

The solvability condition $v^1 b = 0$ yields a unique value for α iff $v^1 F^0_{xx}\langle u_1, u_1\rangle \neq 0$. Then q is the unique solution of (6) satisfying $a\,q = 0$. Finally, β is determined uniquely from the condition

$$a(\beta u_1 + q) = \beta a u_1 = \tau.$$

because $a\,u_1 \neq 0$ by assumption.

Conversely, a regular solution of (1) is a quadratic turning point if in addition $\dim \operatorname{Ker} F^0_x = 1$ and $v^1 F^0_\mu \neq 0$ which is fulfilled in normal case. Besides, the inequality $v^1 F^0_\mu \neq 0$ is much more often fulfilled as the equality $v^1 F^0_\mu = 0$.
Accordingly, for a test on possible turning points, Newton' method may be applied to solve $\Phi_1(z) = 0$. Note however that the result is a necessary condition and the defining properties of a turning point have to be verified after computation.
The second accompanying system is dual to the first system (1) in some way:

$$\boxed{\Phi_2(z) := \Phi_2(\mu, x, v) := \begin{bmatrix} F(\mu, x) \\ v\,F_x(\mu, x) \\ v\,F_\mu(\mu, x) - 1 \end{bmatrix} = 0\,,\ z = (\mu, x, v)\,,\ v \in \mathcal{F}_d}\,. \tag{8}$$

Lemma 2 *Let (μ_0, x_0) be a turning point. Then $\Phi_2(z_0) = \Phi_2(\mu_0, x_0, v^1) = 0$ and*

$$\operatorname{grad} \Phi_2(z_0) : \mathbb{R} \times \mathcal{E} \times \mathcal{F}_d \to \mathcal{F} \times \mathcal{E}_d \times \mathbb{R}$$

has a bounded inverse if and only if (2) does hold.

Proof. Let
$$z = (\sigma, z_1, z^2) \in \mathbb{R} \times \mathcal{E} \times \mathcal{F}_d, \quad g = (g_1, g^2, \tau) \in \mathcal{F} \times \mathcal{E}_d \times \mathbb{R}.$$
Then we have to show again that the linear system

$$[\Phi_2]_z(z_0) z = g\,, \quad z_0 = (\mu_0, x_0, v^1),$$

has a unique solution z for every g. An expansion of this equation yields

$$F^0{}_\mu \sigma + F^0{}_x z_1 = g_1, \tag{9}$$
$$v^1 F^0{}_{x\mu}\sigma + v^1 F^0{}_{xx} z_1 + z^2 F^0_x = g^2, \tag{10}$$
$$z^2 F^0_x = \tau. \tag{11}$$

We use again a suitable decomposition of $\mathcal{E}$ and $\mathcal{F}_d$ and write

$$\begin{aligned} z_1 &= \alpha u_1 + w, \quad v^1 w = 0, \\ z^2 &= \beta v^1 + q, \quad q\,u_1 = 0. \end{aligned} \tag{12}$$

Then we have to show that σ, α, β, w, and q are determined uniquely. From (9) we get

$$\sigma = v^1 g_1$$

because $v^1 F^0{}_x = 0$ and w is the unique solution of

$$F^0{}_x w = g_1 - F^0{}_\mu \sigma, \quad v^1 w = 0,$$

again. Substitution of σ and (12) into (10) yields

$$(\beta v^1 + q){F^0}_x = g^2 - v^1 {F^0}_{x\mu}\sigma - v^1 {F^0}_{xx}(\alpha u_1 + w) =: b. \tag{13}$$

The solvability condition $b\,u_1 = 0$ yields a unique value α iff $v^1 {F^0}_{xx}\langle u_1, u_1\rangle \neq 0$. Then q is the unique solution of (13) satisfying $q\,u_1 = 0$. Finally, β is determined again uniquely from the condition (11),

$$(\beta v^1 + q){F^0}_\mu = \tau,$$

because $v^1 {F^0}_\mu \neq 0$.