Supplements 1 to Chapter V **E. Gekeler**

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To Section 5.1 Let $w(\mu, u) \in \operatorname{Ker} P$ be the unique solution by Theorem 5.4 of

$$H(\mu, u) := (I - Q)F(\mu, u + w(\mu, u)) = 0\,, \tag{1}$$

then derivating w.r.t $\mu \in \mathbb{R}$ yields

$$H_\mu^0 = (I-Q)(F_\mu^0 + F_x^0 w_\mu^0) = (I-Q)F_\mu^0 + F_x^0 w_\mu^0 = 0 \tag{2}$$

with the unique solution $w_\mu^0 \in \operatorname{Ker} P$ by Theorem 5.2. For the F-derivative w.r.t. u with increment $h \in \operatorname{Ker} F_x^0$, we obtain

$$\partial H((\mu_0, u_0); (0, h)) = (I-Q)F_x^0(h + w_u^0 h) = 0\,,\ w_u^0 h \in \operatorname{Ker} P\,,\ h \text{ free}\,.$$

Since $F_x^0 h = 0$ and $(I-Q)F_x^0$ on Ker(P) bijective then $\boxed{w_u^0 = 0}$.

Inserting now $u = U\zeta$ with $\zeta \in \mathbb{R}^\nu$, the bifurcation equatoin is reduced to a mapping $G : \mathbb{R}^{k+\nu} \ni (\mu, \zeta) \mapsto G(\mu, \zeta) \in \mathbb{R}^\kappa$,

$$\boxed{G(\mu, \zeta) := \widetilde{V}^d F(\mu, U\zeta + w(\mu, U\zeta)) = 0\,,\ \ G(\mu_0, \zeta_0) = 0}\,, \tag{3}$$

hence to one or two equation according to whether the problem contains one ore two parameter. In the following formulas, angle brackets denote linear and multilinear applications; for a simple matrix-vector multiplication then $A\langle x\rangle$ instead Ax and e.g.

$$\begin{aligned} F_{xx}^0\langle v_1, v_2\rangle &= \frac{\partial}{\partial\sigma}\frac{\partial}{\partial\tau}F(\mu, x + \sigma v_1 + \tau v_2)\Big|_{\sigma=\tau=0}\Big|_{\mu=\mu_0, x=x_0} \\ F_{\zeta\zeta}^0\langle U, U\rangle &: (\widetilde{\zeta}, \widehat{\zeta}) \mapsto F_{xx}^0\langle U\widetilde{\zeta}, U\widehat{\zeta}\rangle\,. \end{aligned}$$

Derivation of G w.r.t. (μ, ζ) yields, by $w_u^0 = 0$ and $\widetilde{V}^d F_x^0 = 0$,

$$\begin{aligned} \nabla_{(\mu,\zeta)} G^0 &= [\widetilde{V}^d(F_\mu^0 + F_x^0\langle w_\mu^0\rangle)\,,\, \widetilde{V}^d F_x^0\langle U + w_u^0\langle U\rangle\rangle] \\ &= [\widetilde{V}^d F_\mu^0\,,\, \widetilde{V}^d F_x^0\langle U\rangle] = 0 \in \mathbb{R}^\kappa{}_{k+\nu}\,. \end{aligned}$$

Once more derivating yields for the i-th equatin in (3), $i = 1 : \kappa$,

$$\begin{aligned} [G_{\zeta\zeta}^0]^i &= \widetilde{v}^i F_{xx}^0\langle U + w_u^0\langle U\rangle, U + w_u^0\langle U\rangle\rangle + \widetilde{v}^i F_x^0\langle w_{uu}^0\langle U, U\rangle\rangle \\ &= \widetilde{v}^i F_{xx}^0\langle U, U\rangle + \widetilde{v}^i F_x^0\langle w_{uu}^0\langle U, U\rangle\rangle = \widetilde{v}^i F_{xx}\langle U, U\rangle \in \mathbb{R}^\nu{}_\nu \\ [G_{\mu\mu}^0]^i &= \widetilde{v}^i[F_{\mu\mu}^0 + 2F_{x\mu}^0 w_\mu^0 + F_{xx}^0 w_\mu^0 w_\mu^0 + F_x^0 w_{\mu\mu}^0] \\ &= \widetilde{v}^i[F_{\mu\mu}^0 + 2F_{x\mu}^0 w_\mu^0 + F_{xx}^0 w_\mu^0 w_\mu^0] \in \mathbb{R}^k{}_k \\ [G_{\mu\zeta}^0]^i &= \widetilde{v}^i\Big[F_{\mu x}^0\langle U\rangle + F_{\mu x}^0\langle w_u^0\langle U\rangle\rangle\Big] \\ &+ \widetilde{v}^i\Big[F_{xx}\langle U, w_\mu^0\rangle + F_{xx}\langle w_u^0\langle U\rangle, w_\mu^0\rangle + F_x^0\langle\langle U\rangle, w_\mu^0\rangle]\Big] \\ &= \widetilde{v}^i F_{x\mu}^0\langle U\rangle \in \mathbb{R}_\nu\,, \end{aligned}$$

therefore
$$[\nabla^2_{(\mu,\zeta)}G^0]^i = \left[\begin{array}{ll} [G^0_{\mu\mu}]^i & [G^0_{\mu\zeta}]^i \\ [G^0_{\zeta\mu}]^{i,T} & [G^0_{\zeta\zeta}]^i \end{array}\right] \in \mathbb{R}^{k+\nu}{}_{k+\nu}\,, \quad i = 1:\kappa\,.$$

Inserting the solution w^0_μ of (2) one obtains an *explizit representation* of the symmetric HESSE matrix $[\nabla^2_{\mu,\zeta}G^0]^i$, and, for sufficiently smooth G we have the local representation

$$[G(\mu,\zeta)]^i = [G^0]^i + [\nabla_{(\mu,\zeta)}G^0]^i \left[\begin{array}{c}\mu\\ \zeta\end{array}\right] + \frac{1}{2}\,[\mu,\,\zeta]\,[\nabla^2_{(\mu,\zeta)}G^0]^i \left[\begin{array}{c}\mu\\ \zeta\end{array}\right] + \mathcal{O}(\|(\mu,\zeta)\|^3)\,,$$

hence
$$\boxed{[G(\mu,\zeta)]^i = \frac{1}{2}\,[\mu,\,\zeta]\,[\nabla^2_{\mu\zeta}G^0]^i \left[\begin{array}{c}\mu\\ \zeta\end{array}\right] + \mathcal{O}(\|(\mu,\zeta)\|^3)\,,\ i = 1:\kappa}\,. \qquad (4)$$

By a well-known result of MORSE – cf. [M.Hirsch] – terms of higher order can be neglected in this system for qualitative invetigation under the following three conditions:
(1°) The space $\mathcal{E}$ and $\mathcal{F}$ are finite-dimensional,,
(2°) $\kappa = 1$ i.e. one *one* scalar parameter μ is given,
(3°) the matrix $\nabla^2_{\mu\zeta}G^0$ is regular.
If finally the quadratic form (4) describes a *cone*, the existence of branching solutions is guaranteed.

Addenda and Supplements
(1°) Suppose that the dual pairing is defined by a scalar product then

$$vF^0_{xx}(U\zeta, U\zeta) = \zeta^T U^T (vF^0)_{xx} U\zeta\,.$$

(2°) Suppose $x_0 = 0$ and $F(\mu, 0) = 0$ for all $\mu \in \mathbb{R}^k$, then $F^0_\mu = 0\,,\ F^0_{\mu\mu} = 0\,,\ F^0_{\mu x} = 0$, hence also $w^0_\mu = 0$ and thus

$$[\nabla^2_{\mu\zeta}G^0]^i = \left[\begin{array}{ll} 0 & 0 \\ 0 & [G^0_{\zeta\zeta}]^i \end{array}\right] \in \mathbb{R}^{k+\nu}{}_{k+\nu}\,, \quad i = 1:\kappa\,.$$

(3°) It is shown in **(f)** that the bifurcation equation G is odd, $-G(\mu,u) = G(\mu,-u)$, if the mapping F is odd, $-F(\mu,x) = F(\mu,-x)$. Assume that $(\mu_0, x_0) = (0,0)$ is the considered branching point then $G^{0\,i}_\zeta = 0\,,\ G^0_{\zeta\zeta}{}^i = 0$ and $G^{0\,i}_\mu = 0$ hence also $w^0_\mu = 0$. By this way

$$[\nabla^2_{\mu\zeta}G^0]^i = \left[\begin{array}{ll} [G^0_{\mu\mu}]^i & [G^0_{\mu\zeta}]^i \\ [G^0_{\zeta\mu}]^{i,T} & 0 \end{array}\right]\,,\ i = 1:\kappa\,.$$

In this case one needs the the third derivatives

$$\begin{aligned} & G^0_{\zeta\zeta\zeta}{}^i \\ & = \widetilde{v}^i[F^0_{xxx}\langle U + w^0_u\langle U\rangle\rangle^3 + 3F^0_{xx}\langle w^0_{uu}\langle U,U\rangle, U + w^0_u\langle U\rangle\rangle + F^0_x\langle w^0_{uuu}\langle U\rangle^3\rangle \\ & = \widetilde{v}^i[F^0_{xxx}\langle U,U,U\rangle + +3F^0_{xx}\langle w^0_{uu}\langle U,U\rangle, U\rangle]\,. \end{aligned}$$

for investigation of existence and stability. If F^0_{xx} does not vanish, we have to find w^0_{uu}.
(4°) If (2°) and (3°) are both fulfilled then

$$[\nabla^2_{\mu\zeta}G^0]^i = 0\,,\ i = 1:\kappa\,.$$

Lemma 1 *(*MORSE*-Lemma; [M.Hirsch]) Let* $F \in C^r(\mathbb{R}^n;\mathbb{R})$, $r \geq 3$, $F(x^0) = 0$, $\nabla F(x^0) = 0 \in \mathbb{R}_n$, *and let the Hessian* $\nabla^2 F(x_0)$ *be regular with* $0 \leq k \leq n$ *negative eigenvalues. Then there exists an open neighborhood* $0 \in U \subset \mathbb{R}^n$ *and* $x^0 \in V \subset \mathbb{R}^n$ *as well as a diffeomorphism* $\Phi : u \in U \to V$ *with* $\Phi(0) = x^0$ *such that*

$$F(\Phi(u)) = -\sum_{i=1}^{k} u_i^2 + \sum_{i=k+1}^{n} u_i^2 \, .$$