Supplement 4 to Chap. IV

E. Gekeler
05/07/05

Extract from Hartl et al.
(a) Statement of the Problem and Notation Let $\mathcal{X}, \mathcal{U}$ be function spaces over the interval $[0,T] \subset \mathbb{R}$. The control problem (CP) considered by [Hartl] reads:

$$
(1) \qquad
\begin{array}{rcl}
J(x,u,T) & = & S(t,x(T)) + \displaystyle\int_0^T F(t,x(t),u(t),t)dt = \max!,\\
\dot{x}(t) & = & f(t,x(t),u(t))ds,\ t \in [0,T]\,,\ x(0) = x_0\\
0 & \leq & g(t,x(t),u(t)) \in \mathbb{R}^{|g|}\\
0 & \leq & h(t,x(t)) \in \mathbb{R}^{|h|}\\
0 & \leq & a(T,x(T)) \in \mathbb{R}^{|a|}\\
0 & = & b(T,x(T)) \in \mathbb{R}^{|b|}
\end{array}
$$

Every component of g shall depend explicitely on u . For a function $f : (x_1,x_2,x_3) \mapsto f(x_1,x_2,x_3)$ we also write $\nabla_i f(x_1,x_2,x_3) = \nabla_{x_i} f(x_1,x_2,x_3)$, and we write sometimes $f^*[t] = f(t,x^*(t),u^*(t))$ etc.. *Constraint qualification 1:*

$$
\operatorname{rank}\begin{bmatrix} \nabla_x a & \operatorname{diag}(a) \\ \nabla_x b & 0 \end{bmatrix} = |a| + |b|\,.
$$

Constraint qualification 2:

$$
\forall\, t : \operatorname{rank}\ \left[\nabla_u g\,,\ \operatorname{diag}(g)\right] = |g|\,.
$$

This conditions says that $\nabla_u g$ has maximum rank $|g|$.

Definition 1 *Let h be a scalar function,*

$$
\begin{array}{rcccl}
h^0(x,t) & = & h & = & h(x,t),\\
h^1(x,t) & = & \dot{h} & = & \nabla_x h(x,t) f(x,u,t) + h_t(x,t),\\
h^2(x,t) & = & \dot{h}^1 & = & \nabla_x h^1(x,t) f(x,u,t) + h^1_t(x,t),\\
 & & \vdots & & \\
h^p(x,t) & = & \dot{h}^{p-1} & = & \nabla_x h^{p-1}(x,t) f(x,u,t) + h^{p-1}_t(x,t),
\end{array}
$$

The h has the order p if

$$
\nabla_u h^i(x,u,t) = 0,\ i = 0,\dots,p-1,\ \nabla_u h^p(x,u,t) \neq 0.
$$

If h is a vector-valued function then the order of h must be defined elementwise.

Zitat of Hartl et al.:
With respect to the ith constraint $h_i \leq 0$ a subinterval $(\tau_1,\tau_2) \subset [0,T]$ with $\tau_1 < \tau_2$ is called an *interior interval* of a trajectory $x(\cdot)$ if $h_i(x(t),t) < 0$ for all $t \in (\tau_1,\tau_2)$. An interval $[\tau_1,\tau_2]$ with $\tau_1 < \tau_2$ is called a *boundary interval* if $h_i(x(t),t) = 0$ for $t \in [\tau_1,\tau_2]$. An instant τ_1 is called an *entry time* if there is an interior interval ending at $t = \tau_1$ and a boundary interval starting at τ_1. Correspondingly, τ_2 is called an *exit time* if a boundary interval ends and an interior interval starts at τ_2. If the trajectory just touches the boundary at time τ, i.e. $h_i(x(\tau),\tau) = 0$ and if the trajectory is in the interior just before and just after τ, then τ is called a *contact time*. Taken together, entry, exit, and contact times are called *junction times*.

Assumption 1 *(i) Let* $X = C^1_{pc,n}[0,T],\ U = C_{pc,m}[0,T]$*.*
(ii) Let $(x^*, u^*) \in X \times U$ *be a solution with finitely many junction times.*

(iii) Let $l,\ f$ *be continuously F-differentiable in a neighborhood of the set* $\{(x^*(t), u^*(t), t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0,T)\}$*.*
(iv) Let $a,\ b$ *be continuously F-differentiable in a neighborhood of* $(x^*(T), T)$*.*
(v) $\Xi(t) = \{x(t) \in \mathbb{R}^n,\ h(x(t),t) \geq 0 \in \mathbb{R}^q\}$*,* h *sufficiently smooth, see below.*
(vi) $\Omega(t) = \{u(t) \in \mathbb{R}^n,\ g(u(t),t) \geq 0 \in \mathbb{R}^r\}$*,* g *continuously F-differentiable.*
(vii) $\operatorname{rank}[\nabla_u g^*[t], \operatorname{diag} g^*[t]] = r,\ t \in [0,T]$*.*
(viii) $\operatorname{rank}[\nabla_x b^*[T]] = p$*,*
(ix) For every boundary interval $[\tau_1, \tau_2]$ *let*

$$\operatorname{rank}\begin{bmatrix} \nabla_u h_1^{q_1} \\ \vdots \\ \nabla_u h_s^{q_s} \end{bmatrix}^* [t] = s,\ t \in [\tau_1, \tau_2]$$

where $h_i^*[t] = 0,\ i = 1, \ldots, s \leq q,\ h_i^*[t] > 0,\ i = s+1, \ldots q,\ t \in [\tau_1, \tau_2]$*, and* q_i *is the order of* h_i*.*
(x) The linearized differential equation

$$\dot{x} = \nabla_x f(x^*(t), u^*(t), t)x + \nabla_u f(x^*(t), u^*(t), t)u$$

is completely controllable.
(xi) The linearized problem satisfies the SLATER*-condition.*

Notations:

$$\begin{aligned} H(x,u,y,t) &= l(x,u,t) + y^T f(x,u,t), \\ L(x,u,y,v,w,t) &= H(x,u,y,t) + v^T g(x,u,t) + w^T h(x,t). \end{aligned}$$

Theorem 1 *Let assumption 1 be fulfilled. Then there exist*
y^* *piecewise absolutely continuous,*
$v^* \in C_{pc,r}[0,T]$*,*
$w^* \in C_{pc,n}[0,T]$*,*
a vector $c(\tau_i) \in \mathbb{R}^q$ *for each point* τ_i *of discontinuity of* y^**,*
$z^* \in \mathbb{R}^p$*,*
such that
(i)

$$(y^*(t), v^*(t), w^*(t), z^*, c(\tau_1), c(\tau_2), \ldots) \neq (0,0,0,0,\ldots 0)$$

for every $t \in [0,T]$*.*
The following conditions holds a.e. in $[0,T]$*:*

$$u^*(t) = \arg\max_{u \in \Omega(t)} H(x^*(t), u, y^*(t), t),$$

(iii)

$$\nabla_u L^*[t] = \nabla_u H^*[t] + v^*(t)^T \nabla_u g^*[t] = 0,$$

(iv)

$$v^*(t) \geq 0,\ v^*(t)^T g^*[t] = 0,$$

(v)

$$w^*(t) \geq 0,\ w^*(t)^T h^*[t] = 0,$$

(vi)

$$\dot{y}^* = -[\nabla_x L^*[t]]^T.$$

(vii) At terminal time T the following transversality condition holds:

$$y^*(T-) = [\nabla_x a^*[T] + z_1^{*T}\nabla_x b^*[T] + z_2^{*T}\nabla_x h^*[T]]^T,$$
$$z_2^* \geq 0,\ z_2^{*T} h^*[T] = 0.$$

(viii) For any time τ in a boundary interval and for any contact time τ, the costate vector y^* may have a discontinuity given by the following jump conditions:

$$\begin{aligned} y^*(\tau-) &= y^*(\tau+) + c(\tau)h_x^*[\tau], \\ H^*[\tau-] &= H^*[\tau+] - c(\tau)h_t^*[\tau]. \end{aligned}$$
$$c(\tau) \geq 0,\ c(\tau)h^*[\tau] = 0.$$

Assumption 2 *(i) Let $X = W_n^{1,\infty}[0,T],\ U = L_n^\infty[0,T]$.*
(ii) Let $(x^*, u^*) \in X \times U$ be a solution.
(iii) Let $l,\ f$ be continuously F-differentiable in a neighborhood of the set
$\{(x^*(t), u^*(t), t) \in \mathbb{R}^n \times \mathbb{R}^m \times (0,T)\}$.
(iv) Let $a,\ b$ be continuously F-differentiable in a neighborhood of $(x^*(T), T)$.
(v) $\Xi(t) = \{x(t) \in \mathbb{R}^n,\ h(x(t),t) \geq 0 \in \mathbb{R}^q\}$, h sufficiently smooth, see below.
(vi) $\Omega_x(t) = \{u(t) \in \mathbb{R}^n,\ g(x(t),u(t),t) \geq 0 \in \mathbb{R}^r\}$, g continuously F-differentiable.
(vii) $\operatorname{rank}[\nabla_u g^*[t], \operatorname{diag} g^*[t]] = r,\ t \in [0,T]$,
(viii) $\operatorname{rank}[\nabla_x b^*[T]] = p$,
(ix) For every boundary interval $[\tau_1, \tau_2]$ let

$$\operatorname{rank}\begin{bmatrix} \nabla_u h_1^{q_1} \\ \vdots \\ \nabla_u h_s^{q_s} \end{bmatrix}^* [t] = s,\ t \in [\tau_1, \tau_2]$$

where $h_i^*[t] = 0,\ i = 1, \ldots, s \leq q,\ h_i^*[t] > 0,\ i = s+1, \ldots q,\ t \in [\tau_1, \tau_2]$, and q_i is the order of h_i.
(x) Let the linearized differential equation

$$\dot{x} = \nabla_x f(x^*(t), u^*(t), t)x + \nabla_u f(x^*(t), u^*(t), t)u$$

be completely controllable.
(xi) Let the linearized problem satisfy the Slater *condition.*

Theorem 2 *Let assumption 2 be fulfilled. Then there exist*
$y^* \in BV_n[0,T]$,
$v^* \in BV_r[0,T]$,
$w^* \in BV_q[0,T]$, *components w_i nonincreasing, constant on intervals with $h_i^*[t] < 0$,*
$z^* \in \mathbb{R}^p$,
such that
(i)

$$(y^*(t), v^*(t), w^*(T) - w^*(0), z^*) \neq (0,0,0,0),\ t \in [0,T],$$

(ii)

$$u^*(t) = \arg\max_{u \in \Omega(t)} H(x^*(t), u, y^*(t), t),\ \text{a.e. in } [0,T],$$

(iii)
$$\nabla_u L^*[t] = \nabla_u H^*[t] + v^*(t)^T \nabla_u g^*[t] = 0,\ a.e.\ in\ [0,T]$$
(iv)
$$v^*(t) \geq 0,\ v^*(t)^T g^*[t] = 0,\ a.e.\ in\ [0,T],$$
(v)
$$w^*(t) \geq 0,\ w^*(t)^T h^*[t] = 0,\ a.e.\ in \in [0,T]$$
(vi) For all $t_0, t_1 \in [0,T],\ t_0 < t_1$
$$\begin{aligned} y^*(t_1^+) - y^*(t_0^+) &= -\left[\int_{t_0}^{t_1} [\nabla_x H^*[t]dt + v^*(t)^T \nabla_x g^*[t]]dt + \int_{(t_0,t_1]} dw^*(t)^T \nabla_x h^*[t]\right]^T, \\ H^*[t_1^+] - H^*[t_0^+] &= +\int_{t_0}^{t_1} [H_t^*[t] + v^*(t)^T g_t^*[t]]dt - \int_{(t_0,t_1]} dw^*(t)^T h_t^*[t], \end{aligned}$$
(vii)
$$\begin{aligned} & y^*(T-) = [\nabla_x a^*[T] + z_1^{*T} \nabla_x b^*[T] + z_2^{*T} \nabla_x h^*[T]]^T, \\ & z_2^* \geq 0,\ z_2^{*T} h^*[T] = 0. \end{aligned}$$

We may choose $w^*(T) = 0$.
The LAGRANGE function has now the form
$$\begin{aligned} & l((x,u), y, z, v, w) \\ & = [a + z^T b](x(T)) \\ & + \int_0^T l(x,u)dt + \int_0^T [x(t) - x(0) - \int_0^t f(x,u)ds]^T dy(t) \\ & + \int_0^T g(x,u)^T dv(t) + \int_0^T h(x)^T dw(t). \end{aligned}$$
The necessary condition follows from Theorem **??** in the optimum:
$$\begin{aligned} & \nabla_{(x,u)} l((x,u), y, z, v, w)(\delta x, \delta u) = 0, \\ & \int_0^T g(x,u)^T dv(t) = 0,\ \int_0^T h(x)^T dw(t) = 0. \end{aligned}$$
We obtain the two systems of equations
$$\begin{aligned} 0 &= [\nabla[a + z^T b](x(T)) - y^T(T)]\delta x(T) \\ &+ \int_0^T \nabla_x l \delta x dt + \int_0^T [\delta x(t) - x(0) - \int_0^t \nabla_x f \delta x ds]^T dy(t) \\ &+ \int_0^T [\nabla_x g \delta x]^T dv(t) + \int_0^T [\nabla_x h \delta x]^T dw(t), \end{aligned}$$
$$0 = \int_0^T \nabla_u H \delta u dt + \int_0^T [\nabla_u g \delta u]^T dv(t).$$
Thus we obtain the following two systems (**todo**)
$$\begin{aligned} 0 &= \nabla[a + z^T b](x(T)) - y^T(T), \\ 0 &= \nabla_x H + \dot{y}^T + \int_0^T \nabla h_x^T dw(t), \\ 0 &= \nabla_u H + \int_0^T \nabla_u g^T dv(t), \\ 0 &= \int_0^T g(x,u)^T dv(t), \\ 0 &= \int_0^T h(x)^T dw(t). \end{aligned} \tag{2}$$

Theorem 3 *(i) Let assumption 2 be fulfilled.*
(ii) Let the problem 1 be autonomous with possible exception of the mapping g.
(iii) Let (x^*, u^*) *be a solution with finitely many junction times.*
(iv) Let $[\tau_1, \tau_2]$ *be a boundary interval such that* $u^*(t)$ *is in the interior of* $\Omega(t)$ *for* $t \in (\tau_1, \tau_2)$.
(v) Let assumption 2(i) be fulfilled for $t \in [\tau_1, \tau_2]$.
(vi) Let f *and* h *be* $\max\{p_i,\ 1 \le i \le s\} + k$*-times continuously differentiable with* $k > 0$.
Then
y^* *and* w^* *are* $k+1$*-times continuously differentiable in* $[\tau_1, \tau_2]$.

Lemma 1 y^* *is continuous at a junction time* τ *if either (a) or (b) below holds:*
(a) u^* *ist continuous in* τ *and*

$$\operatorname{rank}\begin{bmatrix} \partial g^*[\tau]/\partial u & \operatorname{diag}(g^*[\tau]) & 0 \\ \partial h^{1*}[\tau]/\partial u & 0 & \operatorname{diag}(h^*[\tau]) \end{bmatrix} = q + r.$$

(b) The entry or exit are nontangential, i.e, $h^{1*}[\tau-] < 0$ *or* $h^{1*}[\tau+] > 0$.