Supplements 3 to Chapter IV

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Gradienten Method of Control Theory Cf. [Dyer-McReynolds], Chapter V.

(a) Fixed time interval, no additional constraints

$$\begin{aligned} J(x,u) &= p(x(T),T) + \int_0^T L(x,u,t)dt = \text{Min!}, \\ \dot{x} &= f(x,u,t), \quad x(0) = a, \\ \dot{y} &= -H_x(x,u,t)^T, \quad y(T) = p_x(x(T),T), \\ H(x,y,u,t) &= L(x,u,t) + y^T f(x,u,t). \end{aligned} \tag{1}$$

Let (x,y) be a feasible pair, i.e. let (1)(b),(c) be fulfilled, then

$$\partial J(x,u) = \int_0^T \frac{\partial H}{\partial u} \delta u dt.$$

Let $\operatorname{sgn} x = [\operatorname{sgn} x_1, \dots, \operatorname{sgn} x_n]$ for $x \in \mathbb{R}^n$. Choose

$$\delta u = \varepsilon H_u \text{ oder } \delta u = \varepsilon \operatorname{sgn} H_u$$

where $\varepsilon > 0$ sufficiently small. Let

$$J^j = p(x^j(T),T) + \int_0^T L(x^j,u^j,t)dt$$

Algorithm (a) Gradient Method:

Choose arbitrary nominal control $u^0 : t \mapsto u^0(t)$.

FOR j = 0,1, ...

(1.) Calculate J^j, x^j by solving $\dot{x} = f(x,u)$, $x(0) = a$, $u = u^0$.

(2.) Calculate y^{j+1} by solving $\dot{y} = -H_x(x^j,u^j,y)$, $y(T) = p_x(x^j(T),T)$.

(3.) Calculate u^{j+1} by

$$u^{j+1} = u^j - \varepsilon H_u(x^j,u^j,y^{j+1}),\ \varepsilon \text{ sufficiently small.}$$

(4.) Calculate x^{j+1} by solving $\dot{x} = f(x,u^{j+1},t)$, $x(0) = a$.

(5.) Calculate J^{j+1}.

(6.) If $J^{j+1} < J^j$, goto (2.) where $j := j + 1$.
If $J^{j+1} > J^j$, then reduce ε and got to (3.).
If $J^{j+1} \sim J^j$, then STOP.

(b) Problem (1) with additional constraints

[Teo91] proposes to replace equality and inequality constraints by integral equations in the following way:

$$h(x,u) = 0 \iff \int_0^T \big[h(x,u)\big]^2 dt = 0$$

$$h(x,u) \geq 0 \iff \int_0^T \big[\min\{h(x,u),0\}\big]^2 dt = 0\,.$$

Therefore we introduce r additional control problems with objective functions K_i.

$$K = q(x(T),T) + \int_0^T M(x,u,t)dt = 0 \in \mathbb{R}^r. \tag{2}$$

For $K = [K_i]_{i=1}^r$ let the associated HAMILTON function be

$$\begin{array}{rcl} H_i & = & M_i + y_i^T f,\ i = 1, \ldots, r, \\ \dot{y}_i & = & -[H_i]_x,\ y_i(T) = [q_i]_x(x(T), T). \end{array} \tag{3}$$

The data of the original problem (1)(a)(b) shall be denoted by the index $i = 0$. If (1)(a)(b) and (3) are fulfilled, then

$$\delta K_i = \int_0^T [H_i]_u^T \delta u\, dt.$$

Choose

$$u^{j+1} = u^j - \varepsilon \delta u = u^j + \varepsilon[[H_0]_u + \sum_{k=1}^r z_k [H_k]_u]. \tag{4}$$

Then, in addition, the vector $z = [z_1, \ldots, z_r]$ is to be chosen that $|K|$ decreases. For a nominal solution we have in normal case

$$\delta K_i = \varepsilon \int_0^T [H_i]_u^T [[H_0]_u + \sum_{k=1}^r z_k [H_k]_u] dt \neq 0,\ i = 1, \ldots, r.$$

hence

$$\delta K = b + Qz. \tag{5}$$

This relation is a linear system for the LAGRANGE multipliers z. It has a unique solution if the matrix Q is regular. This represents the controllability condition for the problem. For δK we *choose* the residuum

$$\delta K = K(x^j, u^j).$$

Algorithm (b) Gradient Method
Choose an arbitrary control $u^0 : t \mapsto u^0(t)$.
FOR j = 0,1, ...
(1.) Calculate $J^j,\ x^j$ by solving $\dot{x} = f(x, u),\ x(0) = a,\ u = u^0$.
(2.) Calculate y_0^{j+1} by solving $\dot{y} = -H_x(x^j, u^j, y),\ y_0(T) = p_x(x^j(T), T)$.
Calculate y_k^{j+1} for $k = 1, \ldots, r$ by solving

$$\dot{y} = -[M_k]_x(x^j, u^j, t) - [f^T y]_x(x^j, u^j, t),\ y(T) = [q_k]_x(x^j(T), T).$$

(3.) Calculate residuum of constraints:

$$K(x^j, u^j) = q(x^j(T), T) + \int_0^T M(x^j, u^j, t) dt.$$

Calculate vector z^{j+1} by solving (5) with arguments $x^j, u^j, y_0^{j+1}, y_k^{j+1}$.
Calculate

$$u^{j+1} = u^j - \varepsilon[[H_0]_u + [z^{j+1}]^T [[H_1]_u, \ldots [H_r]_u]^T],\ \varepsilon \text{ sufficiently small.}$$

(4.) Calculate x^{j+1} by solving $\dot{x} = f(x, u^{j+1}, t),\ x(0) = a$.
(5.) Calculate J^{j+1}.
The remaining part is the same as in Algorithm (a).