## Supplement 2 to Chap. IV

**E. Gekeler**
05/07/05

### Proof of Theorem 4.3 and Corollary 4.1

$$
(1)\qquad
\boxed{
\begin{aligned}
J(x,u) &= p(x(0),x(T)) + \int_0^T q(t,x(t),u(t))\,dt = \max!\\
x(t) &= x(0) + \int_0^t f(s,x(s),u(s))\,ds\,,\quad t\in[0\,,\,T]\\
0 &= r(x(0),x(T)) \in \mathbb{R}^{|r|}
\end{aligned}
}
$$

Hamilton function:

$$
H(t,x(t),u(t),y(t)) = q(t,x(t),u(t)) + y(t)f(t,x(t),u(t))\,,\ t\in[0\,,\,T]\,.
$$

**Theorem 1** *(Theorem 4.3) Let $\mathcal{X} = C^1([0,\,T];\mathbb{R}^n)$ , $\mathcal{U} = C^1([0,\,T];\mathbb{R}^m)$ , and let $(x^*,u^*) \in \mathcal{X}\times\mathcal{U}$ be a regular solution of the problem (1). Then there exists a pair $(y^*,z^*) \in \mathcal{X}\times\mathbb{R}_{|r|}$ such that the quadruple $(x^*,u^*,y^*,z^*)$ is solution of the differential-algebraic boundary problem*

$$
(2)\qquad
\boxed{
\begin{array}{rcrll}
\dot x(t) &=& & [\nabla_y H]^T(t,x,u,y) & \in\mathbb{R}^n\\
\dot y(t) &=& - & \nabla_x H(t,x,u,y) & \in\mathbb{R}_n\\
0 &=& & \nabla_u H(t,x,u,y) & \in\mathbb{R}_m\\
0 &=& & r(x(0),x(T)) & \in\mathbb{R}^{|r|}\\
y(0) &=& - & \nabla_1(p+z\,r)(x(0),x(T)) & \in\mathbb{R}_n\\
y(T-) &=& & \nabla_2(p+z\,r)(x(0),x(T)) & \in\mathbb{R}_n
\end{array}
}\ .
$$

Obviously, one may also write the *costate* $y(t)$ as column vector but the present form is advantageous in later computation and in implementation.

*Proof.* The Lagrange function of the problems with equality restrictions reads by § 3.6 and § 12.5

$$
(3)\qquad
\begin{aligned}
L((x,u),y,z) &= [p+z\,r](x(0),x(T))\\
&+ \int_0^T q(x,u)\,dt + \int_0^T dy(t)\left[x(t)-x(0)-\int_0^t f(x,u)\,ds\right]
\end{aligned}
$$

where $z\in\mathbb{R}_{|r|}$ and $y\in \mathrm{NBV}([0,T];\mathbb{R}_n)$. The second integral is a Riemann-Stieltjes integral which counts jumping points of $y$ . The F-derivative of $L$ w.r.t. $(x,u)$ with increment $(\xi,\eta)$ has the form

$$
\begin{aligned}
&\nabla_{(x,u)}L((x,u),y,z)(\xi,\eta)\\
&= \nabla_1[p+z\,r](x(0),x(T))\xi(0) + \nabla_2[p+z\,r](x(0),x(T))\xi(T)\\
&+ \int_0^T [\nabla_x q\,\xi + \nabla_u q\,\eta]dt\\
&+ \int_0^T dy(t)\left[\xi(t)-\xi(0)-\int_0^t (\nabla_x f\,\xi+\nabla_u f\,\eta)\,ds\right].
\end{aligned}
$$

By Corollary 3.4 there exists a pair $(y^*, z^*) \in \mathrm{NBV}(0,T) \times \mathbb{R}_{|r|}$ such that in optimum $(x^*, u^*)$

$$\forall\ (\xi,\eta) \in X \times U : \nabla_{(x,u)} L((x^*,u^*), y^*, z^*)(\xi,\eta) = 0\,.$$

**1. Step** Choose $\eta = 0$ then

$$\begin{aligned} 0 &= \big[\nabla_1[p + z^*\, r](x^*(0), x^*(T))\big]\xi(0) \\ &+ \big[\nabla_2[p + z^*\, r](x^*(0), x^*(T))\big]\xi(T) \\ &+ \int_0^T \nabla_x q\, \xi\, dt + \int_0^T dy^*(t) \left[\xi(t) - \xi(0) - \int_0^t \nabla_x f\, \xi\, ds\right]. \end{aligned} \tag{4}$$

Because of the term $\int_0^T dy^*(t)\xi(t)$ and $\xi \in \mathcal{C}^1$, $y^*$ cannot have a jumping point in the open interval $(0,T)$. Because $y^*(0+) = y^*(0)$ then $y^*$ can at most have a jumping point at $t = T$. Besides, $y^*(T) = 0$ because $y^* \in \mathrm{NBV}(0,T)$. Partial integration yields

$$\begin{aligned} &\int_0^T dy^*(t) \left[\xi(t) - \xi(0) - \int_0^t \nabla_x f\, \xi\, ds\right] \\ &= \big[y^*(T) - y^*(T-)\big] \left[\xi(T) - \xi(0) - \int_0^T \nabla_x f\, \xi\, ds\right] \quad \text{(jumping point)} \\ &+ y^*(T-) \left[\xi(T) - \xi(0) - \int_0^T \nabla_x f\, \xi\, ds\right] \quad \text{(boundary terms)} \\ &- \int_0^{T-} y^*(t)[\dot\xi(t) - \nabla_x f\, \xi]\, dt = -\int_0^{T-} y^* \dot\xi\, dt + \int_0^{T-} y^* \nabla_x f\, \xi\, dt\,. \end{aligned}$$

Inserting into (4) yields with $H = q + y^* f$

$$\begin{aligned} 0 &= \Big[\nabla_1[p + z^*\, r](x^*(0), x^*(T))\Big]\xi(0) + \Big[\nabla_2[p + z^*\, r](x^*(0), x^*(T))\Big]\xi(T) \\ &+ \int_0^{T-} \nabla_x H\, \xi\, dt - \int_0^{T-} y^* \dot\xi\, dt\,. \end{aligned} \tag{5}$$

**2. Step** Choose $\eta = 0$ and $\xi \in C^1$ arbitrary such that $\xi(0) = \xi(T) = 0$, then $y^* \in C^1$ by the augmented fundamental lemma of variational calculus and

$$\dot y^* = -\nabla_x H(x^*, u^*, y^*)\,. \tag{6}$$

Partial integration yields

$$\int_0^{T-} y^* \dot\xi\, dt = y^*(T-)\xi(T) - y^*(0)\xi(0) - \int_0^{T-} \dot y^* \xi\, dt\,.$$

Inserting into (5) yields by using (6)

$$\begin{aligned} 0 &= \Big[\nabla_1[p + z^*\, r](x^*(0), x^*(T))\Big]\xi(0) + \Big[\nabla_2[p + z^*\, r](x^*(0), x^*(T))\Big]\xi(T) \\ &- y^*(T-)\xi(T) + y^*(0)\xi(0)\,. \end{aligned}$$

**3. Step** Choose $\eta = 0$ and $\xi$ arbitrary where either $\xi(T) = 0$ or $\xi(0) = 0$, then

$$\begin{aligned} y^*(0) &= -\nabla_1[p + z\, q](x^*(0), x^*(T))\,, \\ y^*(T-) &= \nabla_2[p + z\, q](x^*(0), x^*(T))\,. \end{aligned}$$

**4. Step** Choose $\xi = 0$ and $\eta \in U$ arbitrary then by using (3)

$$\int_0^T \nabla_u q\,\eta dt - \int_0^T dy^*(t)\left[\int_0^t \nabla_u f\,\eta\,ds\right] = 0\,. \tag{7}$$

Partial integration yields as above

$$\int_0^T dy^*(t)\left[\int_0^t \nabla_u f\,\eta\,ds\right] = -\int_0^{T-} y^*(\nabla_u f\eta)\,dt\,.$$

Therefore we obtain from (7)

$$\int_0^{T-} [\nabla_u q + y^*\nabla_u f]\eta dt = \int_0^{T-} \nabla_u H\eta dt = 0\,.$$

This relation implies by the fundamentallemma that $\nabla_u H = 0$. □

---

**Proof of Theorem 4.6 and Corollary 4.1**

$$\begin{array}{lcll} \mathcal{X} & = & C^1_{pc,n}[0,\,T]\cap C[0,\,T] & \text{the function space of states } x\,, \\ \mathcal{U} & = & C_{pc,m}[0,\,T] & \text{the function space of controls } u\,. \end{array}$$

We seek for a pair $(x,u)\in\mathcal{X}\times\mathcal{U}$ of state $x$ and control $u$ being a solution of the control problem

$$\boxed{\begin{array}{rcl} J(x,u) & = & p(x(0),x(T)) + \displaystyle\int_0^T q(t,x(t),u(t))\,dt = \max! \\ x(t) & = & x(0) + \displaystyle\int_0^t f(s,x(s),u(s))\,ds\,,\quad t\in[0\,,\,T] \\ 0 & = & r(x(0),x(T)) \in \mathbb{R}^{|r|} \\ 0 & \le & g(t,x(t),u(t)) \in \mathbb{R}^{|g|}\,,\quad t\in[0,\,T] \\ 0 & \le & h(t,x(t)) \in \mathbb{R}^{|h|}\,,\quad t\in[0,\,T] \end{array}} \tag{8}$$

where $f,g,h,r,p,q$ are continuously F-differentiable in some suitable domains.
We introduce the scalar HAMILTON function $H$ again, and *two* LAGRANGE functions $\widetilde{L}$ and $L$ are associated to the problem (8):

$$\begin{array}{rcl} H(t,x,u,y) & = & q(t,x,u) + y\,f(t,x,y)\,, \\ L(t,x,u,y,\dot v,\dot w) & = & H(t,x,u,y) + \dot v\,g(t,x,u) + \dot w\,h(t,x)\,, \\ \widetilde{L}((x,u),y,z,v,w) & = & [p + z\,r](x(0),x(T)) \\ & + & \displaystyle\int_0^T q(x,u)\,dt + \int_0^T [dv\,g(x,u) + dw\,h(x)] \\ & + & \displaystyle\int_0^T dy(t)\left[x(t) - x(0) - \int_0^t f(x,u)\,ds\right] \end{array} \tag{9}$$

where $y(t)\in\mathbb{R}_n$, $0\le v(t)\in\mathbb{R}_{|g|}$, $0\le w(t)\in\mathbb{R}_{|h|}$ are row vectors and all integrals over the operational interval $[0,T]$ are again RIEMANN-STIELTJES integrals by § 12.5. Then we obtain a simple analogue to Theorem **??**.

**Theorem 2** *Let the following assumptions be fulfilled:*
(1°) *There exists*

$$(x^*, u^*, y^*, v^*, w^*, z^*) \in \mathcal{X} \times \mathcal{U} \times C^1_{pc,n}[0,\, T] \times C_{pc,|g|}[0,\, T] \times C_{pc,|h|}[0,\, T] \times \mathbb{R}_{|r|}$$

*where* $v^*(t) \geq 0\,,\ w^*(t) \geq 0$ *and*

$$(x^*, u^*) = \arg \max_{(x,u)\in X\times U} \widetilde{L}((x,u), y^*, z^*, v^*, w^*)\,. \tag{10}$$

(2°) *The pair* $(x^*, u^*)$ *satisfies all constraints.*
(3°) $\int_0^T dv^*(t)\, g(t, x^*, u^*) = 0$ *(* complementarity condition *).*
(4°) $\int_0^T dw^*(t)\, h(t, x^*) = 0$ *(* complementarity condition *).*
*Then* $(x^*, u^*)$ *is a solution of (8).*

**Corollary 1** *Adopt the assumptions of Theorem 2.*
*At all points* $t$ *where the six-tuple* $(x^*, u^*, y^*, \dot{v}^*, \dot{w}^*, z^*)$ *continuously exists it is solution of the differential-algebraic boundary value problem*

$$\boxed{\begin{array}{rcll} \dot{x}(t) & = & & [\nabla_y H]^T(t, x, u, y) \\ \dot{y}(t) & = & - & \nabla_x L(t, x, u, y, \dot{v}, \dot{w}) \\ 0 & = & & \nabla_u L(t, x, u, y, \dot{v}, \dot{w}) \\ 0 & = & & r(x(0), x(T)) \\ y(0) & = & - & \nabla_1 (p + z\, r)(x(0), x(T)) \\ y(T-) & = & & \nabla_2 (p + z\, r)(x(0), x(T)) \\ 0 & = & & \dot{v}(t)\, g(t, x, u)\,, \quad t \in [0,\, T] \\ 0 & = & & \dot{w}(t)\, h(t, x)\,, \quad t \in [0,\, T] \\ 0 & \leq & & \dot{v}(t)\,,\ 0 \leq \dot{w}(t)\,, \quad t \in [0,\, T]\,. \end{array}} \tag{11}$$

*Proof.* **(a)** Adopt first that

$$\int_0^T dv(t)[\nabla_x g \xi] = \int_0^T \dot{v}(t)[\nabla_x g \xi]\, dt \tag{12}$$

The LAGRANGE function has now the form

$$\begin{aligned} \widetilde{L}((x,u), y, z, v, w) &= [p + z\, r](x(0), x(T)) \\ &+ \int_0^T q(x,u)\, dt + \int_0^T [dv\, g(x,u) + dw\, h(x)] \\ &+ \int_0^T dy(t) \left[ x(t) - x(0) - \int_0^t f(x,u)\, ds \right] \end{aligned}$$

The necessary condition follows from Theorem 2 in the optimum:

$$\begin{aligned} &\nabla_{(x,u)} L((x,u), y, z)(\xi, \eta) \\ &= \nabla_1 [p + z\, r](x(0), x(T)) \xi(0) + \nabla_2 [p + z\, r](x(0), x(T)) \xi(T) \\ &+ \int_0^T [\nabla_x q\, \xi + \nabla_u q\, \eta] dt \\ &+ \int_0^T dy(t) \left[ \xi(t) - \xi(0) - \int_0^t (\nabla_x f\, \xi + \nabla_u f\, \eta)\, ds \right] \\ &+ \int_0^T dv(t)[\nabla_x g \xi] + \int_0^T dw(t)[\nabla_x h \xi] + \int_0^T dv(t)[\nabla_u g \eta] \end{aligned}$$

$$\int_0^T dv(t)g(x,u) = 0,\ \int_0^T dw(t)h(x) = 0$$

**Step 1.** Choose $\eta = 0$ then in the same way as in the proof of Theorem 4.3, cf. `SUPPLEMENT\chap04b`

$$\begin{aligned}
0 &= \nabla_1[p + z\,r](x(0), x(T))\xi(0) + \nabla_2[p + z\,r](x(0), x(T))\xi(T) \\
&+ \int_0^T \nabla_x q\,\xi\,dt + \int_0^T dv(t)[\nabla_x g\xi] + \int_0^T dw(t)[\nabla_x h\xi] \\
&+ \int_0^T dy(t)\left[\xi(t) - \xi(0) - \int_0^t \nabla_x f\,\xi\,ds\right]
\end{aligned}$$

and then, also in the same way,

$$\begin{aligned}
0 &= \nabla_1[p + z\,r](x(0), x(T))\xi(0) + \nabla_2[p + z\,r](x(0), x(T))\xi(T) \\
&+ \int_0^T \nabla_x q\,\xi\,dt \\
&+ \int_0^{T-} \dot{y}\xi\,dt + \int_0^{T-} y\nabla_x f\xi\,dt + y(0)\xi(0) - y(T-)\xi(T) \\
&+ \int_0^T dv(t)[\nabla_x g\xi] + \int_0^T dw(t)[\nabla_x h\xi]
\end{aligned}$$

**Step 2.** Choose $\eta = 0\,,\ \xi(0) = 0\,,\ \xi(T) = 0$ then

$$0 = \int_0^T \nabla_x q\,\xi\,dt + \int_0^{T-} \dot{y}\xi\,dt + \int_0^{T-} y\nabla_x f\xi\,dt + \int_0^T dv(t)[\nabla_x g\xi] + \int_0^T dw(t)[\nabla_x h\xi]$$

Using (12) we obtain

$$\dot{y} = -\big[\nabla_x q - y\nabla_x f - \dot{v}\nabla_x g - \dot{w}\nabla_x h\big]$$

**Step 3.** Same as in the Proof of Theorem 4.3. Result

$$\begin{aligned}
y(0) &= -\nabla_1[p + z\,q](x(0), x(T))\,, \\
y(T-) &= \nabla_2[p + z\,q](x(0), x(T))\,.
\end{aligned}$$

**Step 4.** Choose $\xi = 0$ and $\eta$ arbitrary then, with the same temporary assumption as above

$$0 = \int_0^T \nabla_u q\,\eta dt + \int \dot{v}\nabla_u g\eta\,dt - \int_0^T dy(t)\left[\int_0^t \nabla_u f\,\eta\,ds\right] = 0 \tag{13}$$

and, after partial integration,

$$0 = \int_0^{T-} \big[\nabla_u q + y^*\nabla_u f + \dot{v}\nabla_u g\big]\eta dt = \int_0^{T-} L_u\eta\,dt\,.$$

**(b)** If one of the dependent variables has a jump at a point $0 < \tau < T$, the above result holds in the corresponding subintervals