Supplements 8 to Chapter III

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28/10/06

Linear Programming, Short Forms

(a) Projection Method for $\max\{a\,x\,;\,Bx \le c\}$,
$a \in \mathbb{R}_n\,,\ B \in \mathbb{R}^m{}_n$ with rows b^i , $\operatorname{rank}(B) = n \le m$.

Initial data: Initial corner x with $\mathcal{A} := \mathcal{A}(x)$ index set of a basis of x and $\mathcal{N} := \mathcal{N}(x)$ index set of the remaining row indices of B. (Succession arbitrary but fixed). Let

$$B^{\mathcal{A}} := [b^k]_{k\in\mathcal{A}(x)}\,,\ B^{\mathcal{N}} := [b^k]_{k\in\mathcal{N}(x)}\,,\ c^{\mathcal{A}} = [\gamma^k]_{k\in\mathcal{A}(x)}\,,\ c^{\mathcal{N}} = [\gamma^k]_{k\in\mathcal{N}(x)}\,,$$

then $B^{\mathcal{A}}$ must be regular and $B^{\mathcal{A}}x = c^{\mathcal{A}}$.

Preparation. Calculate

$$\begin{aligned} &A = [a_k]_{1\le k\le n} := [B^{\mathcal{A}}]^{-1}, && D := B^{\mathcal{N}}A\,,\\ &r := [r^k]_{1\le k\le m-n} := B^{\mathcal{N}}x - c^{\mathcal{N}}\,, && w := a\,A\,,\ \ f := a\,x. \end{aligned}$$

1. Step: (Which search direction a_ν?)
If $w \ge 0$, then STOP (x optimal), else find

$$\begin{aligned} &j := \min\arg_k \min\{w_k\} && \text{(fast)}\\ &j := \min\{k\,;\,y_k < 0\} && \text{(save, B{\scriptsize LAND}'s rule for degenerated corners)} \end{aligned}$$

(a_j new search direction, b^j into basis).

2. Step: (Which step length τ?)
If $d_j := B^{\mathcal{N}}a_j \ge 0$, then STOP (a solution does not exist),
else find

$$\tau =: \min\{r^k/d_j^k\,;\,d_j^k < 0\},\ i = \min\{k\,;\,r^k/d_j^k = \tau\}\,,\ \mu = i + n$$

(b^i out of basis).

3. Step: All data are updated with the *same* exchange step.

$$P = [p^k{}_l] := \begin{bmatrix} A & x\\ D & r\\ w & f \end{bmatrix} \in \mathbb{R}^{m+1}{}_n\,,\quad \begin{pmatrix} \mathcal{A}(x) & \Big| & P\\ \mathcal{N}(x) & & \end{pmatrix} \text{ is the tableau of the method}\,.$$

Compute $Q = [q^k{}_l]$ as follows

$$\begin{aligned} &q^\mu{}_j = 1/p^\mu{}_j && \text{(pivot element)}\,, && q^k{}_j = p^k{}_j/p^\mu{}_j\,,\ k \ne \mu && \text{(pivot column)}\,,\\ &q^\mu{}_l = -p^\mu{}_l/p^\mu{}_j\,,\ l \ne j && \text{(pivot row)}\,, && q^k{}_l = p^k{}_l - p^k_j\,p^\mu{}_l/p^\mu{}_j && \text{(other)}\,. \end{aligned}$$

Form

$$\begin{aligned} \begin{bmatrix} A\\ D \end{bmatrix} &= [q^k{}_l]_{1\le k\le m\,,1\le l\le n}\,,\ x = [q^k{}_{n+1}]_{1\le k\le n}\,,\ r = [q^k{}_{n+1}]_{n+1\le k\le m}\,,\\ w &= [q^{m+1}{}_l]_{1\le l\le m}\,,\ f = q^{m+1}{}_{n+1}\,,\\ \mathcal{A}(x) &= \{\varrho_1,\ldots,\varrho_{j-1}\,,\,\sigma_i\,,\,\varrho_{j+1},\ldots\varrho_n\}\,,\\ \mathcal{N}(x) &= \{\sigma_{n+1},\ldots,\sigma_{i-1}\,,\,\varrho_j\,,\,\sigma_{i+1},\ldots,\sigma_m\} \end{aligned}$$

and return to the first step of the method.

If x is computed in every step as in the present device, updating of the index set $\mathcal{A}(x)$ and $\mathcal{N}(x)$ is only necessary for Bland's rule and for the computation of the Lagrange multipliers y

where $y_{\mathcal{A}} = w$, $y_{\mathcal{N}} = 0$ since only rows are permuted. In the other case the first block row of the tableau can be cancelled whereby the computational amount of work is reduced considerably. In this latter case, μ must be repalced by i and at the end of the iteration the optimal solution x is to be computed by solving the system $B^{\mathcal{A}}x = c^{\mathcal{A}}$.

(b) Computation of an Initial Corner Point in the projection method for $\max\{a\,x\,;\; Bx \leq c\}\,,\; a \in \mathbb{R}_n\,,\; B \in \mathbb{R}^m{}_n\,,\; \operatorname{rank}(B) = n \leq m.$

Initial Data: Feasible point x with $Bx \leq c$, $A = I$ identity matrix, $\mathcal{A}(x) = 0 \in \mathbb{R}^n$.

$$D = [d_k]_{1\leq k\leq n} := A\,,\;\; y = a\,,\;\; r = Bx - c\,,\;\; z = a\,x\,.$$

1. Step: (Search direction.)
If $0 < \mathcal{A}(x) \in \mathbb{R}^n$, then STOP ($x$ is a corner), else find

$$j := \min \arg_k \max\{|y^k|\,,\; [\mathcal{A}(x)]^k = 0\}.$$

If $y^j > 0$, then set $s = -d_j$.
If $y^j < 0$, then set $s = d_j$.
If $y^j = 0$, then set $s = d_j$ where

$$j := \min \arg_k \min\{y_k,\; [\mathcal{A}(x)]^k \neq 0\}\,.$$

2. Step: (Step length as in the projection method with search direction s instead d_j.)
If $\{k \in \{k, \dots m\}\,;\, s^k < 0\} \neq \emptyset$, then find

$$\tau = \min\{f(k) := r^k/s^k\,;\; s^k < 0\}\,,\;\; \mu = \min\{k\,;\; f(k) = \tau\}\,.$$

Set $x := x - \tau\, s$ and go to Step 3.
If $\{k \in \{1, \dots, m\}\,;\, s^k > 0\} \neq \emptyset$, then set $s^k := -s^k$ and go to Step 2.
If $\{k \in \{1, \dots, m\}\,;\, s^k \neq 0\} = \emptyset$, then set $[\mathcal{A}(x)]^j = 0$ and go to Step 1.

3. Step: (Exchange step)

$$P := \begin{bmatrix} D & r \\ y & z \end{bmatrix}\,,$$

compute Q as in the projection method with this matrix P and set $[\mathcal{A}(x)]^j = i$ as well as

$$D = [q^k{}_l]_{1\leq k\leq m\,,1\leq l\leq n}\,,\;\; r = [q^k{}_{n+1}]_{1\leq k\leq m}\,,\;\; y = [q^{m+1}{}_l]_{1\leq l\leq n}\,,\;\; z = q^{m+1}{}_{n+1}\,;$$

go to Step 1.

For the calculation of a feasible point x such that $Bx \leq c$ one considers the problem

$$\min\{\lambda;\; Bx - \lambda e \leq c\,,\; \lambda \geq 0\},\;\; (x\,,\,\lambda) \in \mathbb{R}^{n+1},\;\; e = [1]_{1\leq i\leq m}\,. \tag{1}$$

Let x_0 arbitrary, e.g. $x_0 = 0$, and let $\lambda_0 = \operatorname{Max}\{0\,,\, b^i x_0 - \gamma^i\,,\; i = 1, \dots, m\}$, then $(x_0\,,\,\lambda_0)$ is feasible for (1). Therefore (1) can be solved by the projection method. The optimal solution $(\widetilde{x}\,,\,\widetilde{\lambda})$ exists. $\Omega = \{x \in \mathbb{R}^n\,,\, Bx \leq c\}$ is empty if $\widetilde{\lambda} > 0$ else $\widetilde{x}$ is feasible.

(c) Simplex Algorithm for $\min\{a\,x\,,\;\; Bx = c\,,\, x \geq 0\}$
$B \in \mathbb{R}^m{}_n$ with columns b_j, $\operatorname{rank}(B) = m$, i.e. $m \leq n$.

Initial Data: Initial corner point x (the basis is possibly to be completed),
$\mathcal{A}(x) = (\varrho_1, \dots, \varrho_m)$ index vector resp. index set of the basis variables,

$\mathcal{N}(x) = (\sigma_1, \dots, \sigma_{n-m})$ index vector resp. index set of the non-basis variables (succession arbitrary but fixed).

$$B_{\mathcal{A}} := [b_j]_{j\in\mathcal{A}(x)}\,,\ B_{\mathcal{N}} := [b_j]_{j\in\mathcal{N}(x)}\,,\ a_{\mathcal{A}} := [\alpha_j]_{j\in\mathcal{A}(x)},\ a_{\mathcal{N}} := [\alpha^j]_{j\in\mathcal{N}(x)}.$$

Preparation. Compute

$$\begin{array}{ll} A = [B_{\mathcal{A}}]^{-1}\,,\ D = [d_j]_{1\le j\le n-m} := AB_N, & x := Ac, \\ y = [y_1, \dots, y_{n-m}] = a_{\mathcal{N}} - a_{\mathcal{A}} D, & z := -a\,x. \end{array}$$

1. Step: (Which variable comes into the basis?)
If $y \ge 0$, then x optimal, else find

$$\begin{array}{ll} \nu := \min \arg_k \min\{y_k\} & \text{(fast)},\ (b_\nu \ \text{into the basis}), \\ \nu := \min\{k\,;\, y_k < 0\}, & \text{(save, Bland's rule for degenerated corners)}. \end{array}$$

2. Step: (Which variable is removed from the basis?)
If $d_\nu \le 0$, then STOP (a solution does not exist),
else find

$$\mu := \min \arg_k \min\{x^k/d^k_\nu\,;\, d^k_\nu > 0\},\ \ (a_\mu \text{ out of the basis})\,.$$

3. Schritt: (Exchange Step.)

$$\begin{array}{ll} \text{Set } P = [p^k{}_l] := \begin{bmatrix} D & x \\ y & z \end{bmatrix}, & \text{compute } Q = [q_{k\,l}] \ \text{ as follows} \\ q^\mu{}_\nu = 1/p^\mu{}_\nu \text{ (pivot element)}, & q^k{}_\nu = -p^k_\nu/p^\mu_\nu,\ k \ne \mu \text{ (pivot column)}, \\ q^\mu{}_l = p^\mu{}_l/p^\mu{}_\nu\,,\ l \ne \nu \text{ (pivot row)}, & q^k{}_l = p^k{}_l - p^k{}_\nu\, p^\mu{}_l/p^\mu{}_\nu \text{ (other)}\,. \end{array}$$

Set

$$\begin{array}{ll} D = [q^k{}_l]_{1\le k\le m\,,\,1\le l\le n-m}, & x = [q^k{}_{n-m+1}]_{1\le k\le m}\,, \\ y = [q^{m+1}{}_l]_{1\le l\le n-m}, & z = q^{m+1}_{n-m+1}\,; \\ \mathcal{A}(x) = (\varrho_1, \dots, \varrho_{\mu-1}\,,\ \sigma_\nu\,,\ \varrho_{\mu+1}, \dots, \varrho_m)\,, & \\ \mathcal{N}(x) = (\sigma_1, \dots, \sigma_{\nu-1}\,,\ \varrho_\mu\,,\ \sigma_{\nu+1}, \dots, \sigma_{n-m}) & \end{array}$$

and return to the first step of the method.
Using the matrix P,

$$\begin{array}{c|c} & \mathcal{N}(x) \\ \hline \mathcal{A}(x) & P \end{array}$$

is called simplex tableau, x contains the current values of the basis variables, z the *negative* current value of the cost functional.

$\mathcal{A}(x)$ and $\mathcal{N}(x)$ must be stored up since the columns of B are permuted.