Supplements 4 to Chapter III

E. Gekeler
28/10/06

Global Lagrange Theory

(a) Formulation of the Problem Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be real normed vector spaces, let $\emptyset \neq \mathcal{C} \subset \mathcal{X}$ be an arbitrary set being not necessarily open or a subspace, and let $\emptyset \neq \mathcal{K} \subset \mathcal{Y}$ be an order cone with dual cone $\mathcal{K}_d$ in the dual space $\mathcal{Y}_d$ of $\mathcal{Y}$, cf. § 1.10. Further, let

$$f : \mathcal{C} \to \mathbb{R}\,, \quad g : \mathcal{C} \to \mathcal{Y}\,, \quad h : \mathcal{C} \to \mathcal{Z}\,,$$

be three mappings and consider the general *minimum problem* (MP)

$$\{f(x)\,;\; x \in \mathcal{C}\,,\; -g(x) \in \mathcal{K}\,,\; h(x) = 0\} = \min! \tag{1}$$

with feasible set $\mathcal{S} = \{x \in \mathcal{C}\,,\; -g(x) \in \mathcal{K}\,,\; h(x) = 0\}$ and LAGRANGE function

$$L : \mathcal{C} \times \mathcal{Y}_d \times \mathcal{Z}_d \ni (x, y, z) \mapsto L(x, y, z) = f(x) \pm y \circ g(x) + z \circ h(x) \in \mathbb{R}\,.$$

The sign of y is positive in the present minimum problem and negative in the corresponding maximum problem; cf. § 3.2**(b)**. The problem (1) is called *convex* again if $\mathcal{C}$, f convex, g $\mathcal{K}$-convex, and h *affine linear*. For $-g(x) \in \mathcal{K}$ we write briefly $g(x) \leq 0$ and observe that, in the present situation, the LAGRANGE multipliers y and z are elements of the dual spaces $\mathcal{Y}_d$ and $\mathcal{Z}_d$, resp. may be canonically identified with elements of these spaces. Altogether we are faced with the following constellation:

mapping:	f	g	h
range:	$\mathbb{R}$	$\mathcal{Y}$	$\mathcal{Z}$
order cone:	$\mathbb{R}_{\geq 0}$	$\mathcal{K}$	$\mathcal{L} = \{0\}$
dual elements:	$\varrho \in \mathbb{R}$	$y \in \mathcal{Y}_d$	$z \in \mathcal{Z}_d$

.

The equality restrictions $h(x) = 0$ may *not* be replaced by double inequalities because we have to suppose sometimes that the interior $\mathrm{int}(\mathcal{K})$ of $\mathcal{K}$ is not empty. In slight generalization, the restrictions differ from each other by the two order cones $\mathcal{K}$ and $\mathcal{L}$ according to

$$g(x) \overset{\mathcal{K}}{\leq} 0\,,\; \mathrm{int}(\mathcal{K}) \neq \emptyset\,, \quad h(x) \overset{\mathcal{L}}{\leq} 0\,,\; \mathrm{int}(\mathcal{L}) = \emptyset\,.$$

Definition 1 (1°) *The minimum problem (MP) is called* G-, H-, F-differentiable *if the mappings f, g, h are* GATEAUX-, HADAMARD-, *resp.* FRÉCHET-*differentiable in an open superset of $\mathcal{C}$.*
(2°) *The problem (MP) is* convex *if $\mathcal{C} \subset \mathcal{X}$ is a convex set, $f : \mathcal{C} \to \mathbb{R}$ a convex mapping, $g : \mathcal{C} \to \mathcal{Y}$ a $\mathcal{K}$-convex mapping, and $h : \mathcal{C} \to \mathcal{Z}$ a* affine linear *mapping.*

The fundamental Theorem 3.2 now reads:

Theorem 1 *Let $(x^*, y^*, z^*) \in \mathcal{C} \times \mathcal{K}_d \times \mathcal{Z}_d$ be a triple such that*

$$x^* = \arg \begin{matrix}\min\\ \max\end{matrix} \{f(x^*) \pm y^* \circ g(x^*) + z^* \circ h(x^*)\,,\; x \in \mathcal{C}\}\,, \tag{2}$$

and let $x^* \in \mathcal{S}$ as well as $y^* \circ g(x^*) = 0$. Then

$$x^* = \arg \begin{matrix}\min\\ \max\end{matrix} \{f(x)\,,\; x \in \mathcal{C}\,,\; g(x) \leq 0\,,\; h(x) = 0\}\,.$$

Proof. For all $x \in \mathcal{S}$

$$f(x^*) \;=\; f(x^*) \pm y^* \circ g(x^*) + z^* \circ h(x^*) \;\underset{\geq}{\leq}\; f(x) \pm y^* \circ g(x) + z^* \circ h(x) \;\underset{\geq}{\leq}\; f(x).$$

□

Accordingly the candidates for maximum and minimum points can be found by (2) at the same time.

Let the extremal problem be F-differentiable then (2) yields the necessary condition

$$\begin{array}{ll} \forall\, x \in \mathcal{C} : \nabla_x L(x^*, y^*, z^*)(x - x^*) \geq 0 & \text{(minimum problem)}, \\ \forall\, x \in \mathcal{C} : \nabla_x L(x^*, y^*, z^*)(x - x^*) \leq 0 & \text{(maximum problem)}, \\ \forall\, x \in \mathcal{C} : \nabla_x L(x^*, y^*, z^*)(x - x^*) = 0 & \text{(if } \mathcal{C} \text{ open in } \mathcal{X}\text{, e.g. } \mathcal{C} = \mathcal{X}). \end{array} \tag{3}$$

The condition (3) is sufficient for (2) if the minimum problem is convex resp. the maximum problem is concave since the associated LAGRANGE function L then is in x convex resp. concave on the convex set $\mathcal{C}$; cf. Lemma 1.25.

(b) (Primal) Lagrange Problem (LP)

Find a triple $(x^*, y^*, z^*) \in \mathcal{C} \times \mathcal{K}_d \times \mathcal{Z}_d$ so that

$$\boxed{(x^*, y^*, z^*) = \arg\min_{x\in\mathcal{C}} \sup\{L(x, y, z)\,,\ y \in \mathcal{K}_d\,,\ z \in \mathcal{Z}_d\}}\ , \tag{4}$$

Let us first summarize some properties of the LAGRANGE function:

Lemma 1 *(Lemma 3.3)* (1°) *Let $x \in \mathcal{S}$ and $y^* \in \mathcal{K}_d$ so that $y^* \circ g(x) = 0$ (e.g. $y^* = 0$). Then*

$$\forall\, z \in \mathcal{Z}_d : f(x) = L(x, y^*, z) = \max\{L(x, y, z)\,,\ y \in \mathcal{K}_d\}.$$

(2°) *Let the order cone $\mathcal{K}$ in $\mathcal{Y}$ be closed and let conversely $\max_{(y,z)\in\mathcal{K}_d\times\mathcal{Z}_d} L(x, y, z)$ exist for some $x \in \mathcal{C}$. In other words let, for some $x \in \mathcal{C}$,*

$$\exists\, (y^*, z^*) \in \mathcal{K}_d \times \mathcal{Z}_d : L(x, y^*, z^*) = \max\{L(x, y, z)\,,\ y \in \mathcal{K}_d\,,\ z \in \mathcal{Z}_d\}\,,$$

then x is feasible, $x \in \mathcal{S}$, and moreover $y^* \circ g(x) = 0$.

(3°)

$$\mathcal{S} = \emptyset \iff \forall\, x \in \mathcal{C} : \sup\{L(x, y, z)\,,\ y \in \mathcal{K}_d\,,\ z \in \mathcal{Z}_d\} = \infty\,.$$

Proof. (See [Krabs], § 2.1 for $h = 0$).

(1°) Let $x \in \mathcal{S}$ then $g(x) \leq 0$ and $h(x) = 0$, hence

$$\forall\, y \in \mathcal{K}_d\ \ \forall\, z \in Z_d : y \circ g(x) \leq 0 \ \text{ and } \ z \circ h(x) = 0\,,$$

which proves the assertion.

(2°) By assumption, for $x \in \mathcal{C}$,

$$\forall\, y \in \mathcal{K}_d\ \ \forall\, z \in Z_d : y^* \circ g(x) + z^* \circ h(x) \geq y \circ g(x) + z \circ h(x)\,. \tag{5}$$

(2.1°) Let $0 = z \in \mathcal{Z}_d$. By (5) we have $y^* \circ g(x^*) \geq 0$ for $y = 0$. But, on the other side, $y \circ g(x) \leq 0$ for all $y \in \mathcal{K}_d$, else we have a contradiction to (4) since with $y \in \mathcal{K}_d$ also $\alpha\, y \in \mathcal{K}_d$ for all $\alpha \geq 0$. Therefore $g(x) \leq 0$ by the cone corollary, since $\mathcal{K}$ closed. Setting $y = y^*$ we

obtain together $y^* \circ g(x) = 0$.
(2.2°) Let $0 = y \in \mathcal{K}_d$ then $h(x) = 0$ because $z \in Z_d$ arbitrary. (If $h(x) \overset{\mathcal{L}}{\leq} 0$ with a closed cone $\mathcal{L}$, the assertion follows in the same way as in (2.1°). Altogether x is feasible.
(3°) Let $\mathcal{S}$ be non-empty then by (1°) for $x \in \mathcal{S}$

$$f(x) = \sup_{(y,z)\in\mathcal{K}_d\times Z_d} L(x,y,z) < \infty.$$

If $\mathcal{S}$ empty and $x \in \mathcal{C}$, $-g(x) \notin \mathcal{K}$ or $h(x) \neq 0$.
(3.1°) Let $-g(x) \notin \mathcal{K}$ then, by Lemma 1.20, there exists a $y^* \in \mathcal{K}_d$ such that $y^* \circ g(x) > 0$. Since $\alpha y^* \in \mathcal{K}_d$ for $\alpha \geq 0$ we obtain $\sup_{y\in\mathcal{K}_d} L(x,y,z) = \infty$.
(3.2°) Let $h(x) \neq 0$ then $\sup_{z\in Z_d} L(x,y,z) = \infty$.
□

Corollary 1 *Let the order cone $\mathcal{K} \subset Y$ be closed and $x \in \mathcal{C}$. Then x feasible, $x \in \mathcal{S}$, and $y^* \circ g(x) = 0$ if and only if*

$$L(x,y^*,z^*) = \mathrm{Max}_{(y,z)\in\mathcal{K}_d\times Z_d} L(x,y,z)\,. \tag{6}$$

Proof. Let (6) then the assertion follows from Lemma 1(2°); the other direction follows from Lemma 1(1°).

Theorem 2 *(Theorem 3.12,* Lagrange *problem sufficient) Let the order cone $\mathcal{K} \subset \mathcal{Y}$ be closed and let $(x^*,y^*,z^*) \in \mathcal{C}\times\mathcal{K}_d\times\mathcal{Z}_d$ be a solution of (LP) then x^* is a solution of the minimum problem (MP).*

Proof. Let $(x^*,y^*,z^*))$ be a solution of (LP) then $x^* \in \mathcal{C}$ and

$$L(x^*,y^*,z^*) = \max{}_{y\in\mathcal{K}_d, z\in\mathcal{Z}_d} L(x^*,y,z)$$

($\sup_{y\in\mathcal{K}_d, z\in\mathcal{Z}_d} L(x,y,z)$ is not necessarily attained for all $x \in \mathcal{C}$ but by assumption for x^*). Then x^* feasible by Lemma 1(2°) and $y^* \circ g(x^*) = 0$. By Lemma 1(1°),

$$(L(x^*,y^*,z^*) =)\ f(x^*) = \min{}_{x\in\mathcal{C}} \sup{}_{y\in\mathcal{K}_d, z\in\mathcal{Z}_d} L(x,y,z)\ \ \square.$$

Lemma 2 *Let $\mathcal{K}$ be closed. Then the minimum problem (MP) (1) and the* Lagrange *problem (LP) (3) are equivalent in the sense that if x^* solution of (MP) and $(\widetilde{x}^*,y^*,z^*)$ solution of (LP), then $x^* = \widetilde{x}^*$.*

Note that this result is an *equivalence theorem* which says nothing about the *existence* of the resp. solutions.
Proof. (1°) Let x^* be a solution of (MP). Then

$$\begin{aligned} f(x^*) &:= \min{}_{x\in\mathcal{C}}\{f(x)\,,\ g(x) \leq 0\,,\ h(x) = 0\} \\ &= \min{}_{x\in\mathcal{C}, g(x)\leq 0, h(x)=0} \sup{}_{y\geq 0, z}\{f(x) + y\circ g(x) + z\circ h(x)\}, \\ &= \min{}_{x\in\mathcal{C}} \sup\{L(x,y,z)\,,\ y \in \mathcal{K}_d\,,\ z \in \mathcal{Z}_d\}\ \text{(Lemma 1(1°))} \\ &=: L(x^*,y^*,z^*)\,. \end{aligned} \tag{7}$$

Therefore (x^*,y^*,z^*) is solution of (LP).
(2°) (Same as proof of Theorem 2.) Let (x^*,y^*,z^*) be a solution of (LP) then $y^* \circ g(x^*) = 0$

and x^* feasible by Lemma 1(2°). Also $f(x^*) = L(x^*, y^*, z^*)$ by Lemma 1(1°), and the same conclusions of (6) hold in converse direction therefore x^* is solution of (MP). □

(c) Saddlepoint Problems In the following both results let $\mathcal{A}$, $\mathcal{B}$ be arbitrary sets and

$$f : \mathcal{A} \times \mathcal{B} \ni (x, y) \mapsto f(x, y) \in \mathbb{R}$$

an arbitrary function.

Definition 2 *A pair $(x^*, y^*) \in \mathcal{A} \times \mathcal{B}$ is a saddlepoint of f if*

$$\forall\, x \in \mathcal{A},\ \forall\, y \in \mathcal{B} :\ f(x^*, y) \le f(x^*, y^*) \le f(x, y^*). \tag{8}$$

Lemma 3 *The function f has a saddlepoint in $\mathcal{A} \times \mathcal{B}$ if and only if*

$$\max{}_{y\in\mathcal{B}} \inf{}_{x\in\mathcal{A}} f(x, y) = \min{}_{x\in\mathcal{A}} \sup{}_{y\in\mathcal{B}} f(x, y). \tag{9}$$

Here, max instead sup says that the supremum is actually attained.

Proof. [Ekeland].

To the minimum problem (1) we associate the following *saddlepoint problem* (SPP) where again $\mathcal{K}_d \subset \mathcal{Y}$ is the order cone:

Find a *saddlepoint* $(x^*, y^*, z^*) \in \mathcal{C} \times \mathcal{K}_d \times \mathcal{Z}_d$ such that

$$\forall\, (x, y, z) \in \mathcal{C} \times \mathcal{K}_d \times \mathcal{Z}' :\ L(x^*, y, z) \le L(x^*, y^*, z^*) \le L(x, y^*, z^*). \tag{10}$$

Theorem 3 *((SPP) sufficient.) Let the order cone $\mathcal{K}$ be closed. If (x^*, y^*, z^*) is a saddle point, i.e. a solution of (10), then x^* is a minimum point, i.e. a solution of (1).*

Proof. (a1) For $z = z^*$ we obtain from the left inequality in (10)

$$y \circ g(x^*) \le y^* \circ g(x^*). \tag{11}$$

But $y + y^* \in \mathcal{K}_d$ for all $y \in \mathcal{K}_d$ because $\mathcal{K}_d$ is convex. Writing $y + y^*$ for y in (11) we obtain

$$\forall\, y \in \mathcal{K}_d : y \circ g(x^*) \le 0.$$

By the Cone Corollary we thus obtain $g(x^*) \le 0$ which implies $y^* \circ g(x^*) \le 0$. Setting $y = 0$ in (11) we obtain $y^* \circ g(x^*) \ge 0$ hence together $y^* \circ g(x^*) = 0$.

(a2) For $y = y^*$ in the left inequality of (10) we obtain $(z - z^*) \circ h(x^*) \le 0$. As $z - z^* \in Z_d$ is arbitrary we then find that $h(x^*) = 0$. Thus x^* is a feasible point.

(b) For feasible x we obtain from the right inequality in (10) using (a)

$$f(x^*) + 0 \le f(x) + y^* \circ g(x) \le f(x)$$

because $y^* \in \mathcal{K}_d$. Hence x^* is a minimum point.□

(d) Dual Lagrange Problem (DLP) Find a triple $(x^*, y^*, z^*) \in \mathcal{X} \times \mathcal{K}_d \times \mathcal{Z}_d$ such that

$$L(x^*, y^*, z^*) = \max{}_{(y,z)\in\mathcal{K}_d\times\mathcal{Z}_d} \inf{}_{x\in\mathcal{C}} L(x, y, z)\,. \tag{12}$$

Corollary 2 *(Equivalence) Let the order cone $\mathcal{K} \in \mathcal{Y}$ be closed and let the* Lagrange *function L have a saddlepoint. Then the primal and the dual* Lagrange *problem (LP) and (DLP) are equivalent,*

$$L(x^* y^*, z^*) = \arg\min{}_{x\in\mathcal{C}} \sup{}_{(y,z)\in\mathcal{K}_d\times\mathcal{Z}_d} L(x, y, z) = \max{}_{(y,z)\in\mathcal{K}_d\times\mathcal{Z}_d} \inf{}_{x\in\mathcal{C}} L(x, y, z)\,,$$

and $L(x^*, y^*, z^*) = f(x^*)$ if $y^* \circ g(x^*) = 0$.

(e) Necessary Conditions The proof of the *existence* of the above used LAGRANGE multipliers is much more difficult and needs besides the convexity of the problem also further regularity assumptions on the equality constraints $h(x) = 0$.

Definition 3 *Let the vector space $\mathcal{Y}$ be normed, $\mathcal{K} \subset \mathcal{Y}$ the (convex) order cone with non-empty interior,* $\text{int}(\mathcal{K}) \neq \emptyset$ *, and let $g : \mathcal{C} \to Y$ $\mathcal{K}$-konvex.*
(1°) *g satisfies the* SLATER *condition (S) if*

$$\mathcal{A} := \{x \in \mathcal{C},\ g(x) < 0\} \neq \emptyset.$$

(2°) *g satisfies the* KARLIN *condition (K) if*

$$\mathcal{B} := \{y \in \mathcal{K}_d : y \circ g(\mathcal{C}) \subset \mathbb{R}_{\geq 0}\} = \{0\}.$$

or, in other words,

$$\forall\, 0 \neq y \in \mathcal{K}_d\ \ \exists\, x \in \mathcal{C} : y \circ g(x) < 0.$$

$\neg$(S) is the condition $\mathcal{A} = \emptyset$ and $\neg$(K) is the condition $\mathcal{B} \neq \{0\}$. The following theorem says that either (S) holds or $\neg$(K) but never both. Therefore it is sometimes called the *basic alternative theorem.*

Theorem 4 *Adopt the assumption of definition 3 then the* SLATER *condition (S) and the* KARLIN *condition (K) are equivalent.*

Proof. Cf. e.g. [Craven78], § 2.5.
(1°) Let $x \in \mathcal{A}$ and $y \in \mathcal{B}$. From $g(x) < 0$ and $y \in \mathcal{K}_d$ we then obtain by the Cone Corollarry that $y = 0$.
(2°) We prove $\neg(S) \Longrightarrow \neg(K)$. Let $\mathcal{W} \subset Y$ be defined by

$$\mathcal{W} = g(\mathcal{C}) + \text{int}(\mathcal{K}).$$

(i) $\mathcal{W}$ is open:

$$x \in \mathcal{C},\ k \in \text{int}(\mathcal{K}) \Longrightarrow w = g(x) + k \in \mathcal{W}.$$

Because $\text{int}(\mathcal{K}) \neq \emptyset$, there exists an open ball $\mathcal{N}$ such that $k + \mathcal{N} \subset \text{int}(\mathcal{K})$ hence $w + \mathcal{N} \subset \mathcal{W}$. Therefore $\mathcal{W}$ is open.
(ii) $\mathcal{W}$ is convex: Let $w_i = g(x_i) + k_i \in \mathcal{W},\ x_i \in \mathcal{C},\ k_i \in \text{int}(\mathcal{K}),\ i = 1, 2$, let $0 < \lambda < 1$ and $x = \lambda x_1 + (1 - \lambda)x_2$. Then $x \in \mathcal{C}$ because $\mathcal{C}$ is convex. Let

$$\begin{aligned} u &:= (1-\lambda)w_1 + \lambda w_2 \\ &= (1-\lambda)(w_1 - g(x_1)) + \lambda(w_2 - g(x_2)) \\ &+ [(1-\lambda)g(x_1) + \lambda g(x_2) - g(\lambda x_1 + (1-\lambda)x_2)] + g(x) \end{aligned}$$

then $[\ldots] \geq 0$ hence $[\ldots] \in \mathcal{K}$ because g is $\mathcal{K}$-convex, and

$$\begin{aligned} &(1-\lambda)(w_1 - g(x_1)) = (1-\lambda)k_1 \in \text{int}(\mathcal{K}), \\ &\lambda(w_2 - g(x_2)) = \lambda k_2 \in \text{int}(\mathcal{K}). \end{aligned}$$

So we obtain

$$u \in \text{int}(\mathcal{K}) + \text{int}(\mathcal{K}) + \mathcal{K} + g(x) = \text{int}(\mathcal{K}) + g(x)$$

hence $u = (1-\lambda)w_1 + \lambda w_2 \in \mathcal{W}$ and therefore $\mathcal{W}$ is a convex set.
(iii) $0 \notin \mathcal{W}$: From $\neg(S)$ we obtain $\{x \in \mathcal{C},\ g(x) < 0\} = \emptyset$. Let $0 \in \mathcal{W}$, then there exist $x \in \mathcal{C}$ and $k \in \text{int}(\mathcal{K})$ with $0 = g(x) + k$. This equation yields $k = -g(x) > 0$ hence $g(x) < 0$ in

contradiction to assumption.
(iv) Applying the Separation Theorem to $\mathcal{C} = \mathcal{W}$ and $\mathcal{D} = \{0\}$, there exists a $0 \neq y \in Y'$ such that

$$0 = y(0) \leq \inf_{w \in \mathcal{W}} y(w) \leq y(w), \; w \in \mathcal{W}.$$

Hence there exists a $0 \neq y \in Y_d$ with $y(\mathcal{W}) \subset \mathbb{R}_+$.
(v) $y \in \mathcal{K}_d$: Let $x_0 \in \mathcal{C}$ and $k \in \mathrm{int}(\mathcal{K})$. Then there exists an open ball $\mathcal{N}$ containing zero such that $k + \mathcal{N} \subset \mathrm{int}(\mathcal{K})$. For λ sufficiently large we obtain $\lambda^{-1} g(x_0) \in \mathcal{N}$ hence

$$k - \lambda^{-1} g(x_0) \in \mathrm{int}(\mathcal{K}) \Longrightarrow \lambda(k - \lambda^{-1} g(x_0)) \in \mathrm{int}(\mathcal{K}),$$

because $\mathcal{K}$ is a cone. Therefore $p = \lambda k - g(x_0) \in \mathrm{int}(\mathcal{K})$ if λ is sufficiently large which yields $\lambda k = g(x_0) + p \in \mathcal{W}$ and thus

$$\forall \; k \in \mathrm{int}(\mathcal{K}) : \lambda^{-1} y(\lambda k) = y(k) \geq 0,$$

by (iv). As y is continuous we also have $y(k) \geq 0$ for all $k \in \mathcal{K} \subset \overline{\mathrm{int}(\mathcal{K})}$. Therefore $y \in \mathcal{K}_d$.
(vi) The above defined $0 \neq y \in \mathcal{K}_d$ satisfies $y \circ g(\mathcal{C}) \subset \mathbb{R}_+$ hence $\neg(K)$: For $x \in \mathcal{C}$ and $k \in \mathrm{int}(\mathcal{K})$ we obtain $w = g(x) + \varepsilon k \in \mathcal{W}$ for all $\varepsilon > 0$ by definition of $\mathcal{W}$. Therefore

$$y(g(x)) = y(w) - \varepsilon y(k) \geq 0 - \varepsilon y(k) \to 0, \; \varepsilon \to 0.$$

We thus have $y \circ g(x) \geq 0$ for all $x \in \mathcal{C}$. □

The assumption $\mathrm{int}(\mathcal{K}) \neq \emptyset$ implies that $\mathrm{int}(\mathcal{W}) \neq \emptyset$ and thus Theorem 4 can also be proved by application of EIDELHEIT's separation theorem.

By applying the basic alternative theorem we now show that the existence of a saddlepoint is also necessary for the existence of a solution of the minimum problem (1) in the case where the problem is convex and no equality restrictions $h(x) = 0$ occur:

Theorem 5 *((SPP) necessary, $h = 0$) Let the minimum problem (1) be convex and $h \equiv 0$. Suppose that*
(1°) $\mathrm{int}(\mathcal{K}) \neq \emptyset$,
(2°) g satisfies the KARLIN *condition.*
If

$$x^* = \arg\min\{f(x), \; x \in \mathcal{C}, \; g(x) \leq 0\}$$

then

$$\exists \; 0 \neq y^*, \; \forall \; (x, y) \in \mathcal{C} \times \mathcal{K}_d : L(x^*, y) \leq L(x^*, y^*) \leq L(x, y^*). \tag{13}$$

where $L(x, y) = f(x) + y \circ g(x)$, and moreover $y^* \circ g(x^*) = 0$.

Proof. Cf. e.g. [Craven78], § 2.5. The set

$$\mathcal{P} := \mathbb{R}_{\geq 0} \times \mathcal{K} \subset \mathbb{R} \times Y$$

defines a convex cone with $\emptyset \neq \mathrm{int}(\mathcal{P}) \subset \mathrm{int}(\mathbb{R}_{\geq 0}) \times \mathcal{K}$. Moreover, we have $f(x) \geq f(x^*)$ for every feasible x hence $-(f(x) - f(x^*)) \notin \mathrm{int}(\mathbb{R}_{\geq 0})$. Let the mapping G be defined by

$$G : \mathcal{C} \supset x \mapsto (f(x) - f(x^*)), g(x))$$

then G is $\mathcal{P}$-convex and $\neg$ SLATER holds for G:

$$\{x \in \mathcal{C} : -G(x) \in \mathrm{int}(\mathcal{P})\} = \emptyset.$$

Therefore also ¬ KARLIN is fulfilled for G, i.e., there exists a $(0,0) \neq (\varrho^*, y^*) \in \mathcal{P}_d = \mathbb{R}_{\geq 0} \times \mathcal{K}_d$ such that

$$\forall\, x \in \mathcal{C} : \varrho^*(f(x) - f(x^*)) + y^* \circ g(x) \geq 0. \tag{14}$$

If $\varrho^* = 0$ then $y^* \neq 0$ and (14) yields

$$\forall\, x \in \mathcal{C} : y^* \circ g(x) \geq 0.$$

which contradicts the KARLIN condition. Therefore $\varrho^* \neq 0$ and we may write $\varrho^* = 1$ after possible division by ϱ^*. If the complement condition $y^* \circ g(x^*) = 0$ holds then (14) yields the right inequality of the (SPP) (13). For $x = x^*$ however we obtain from (14) that $y^* \circ g(x^*) \geq 0$. But $g(x^*) \leq 0$ and $y^* \in \mathcal{K}_d$ hence $y^* \circ g(x^*) \leq 0$ and the complement condition is fulfilled. The left inequality of the (SPP) (13) holds, too, because $y \circ g(x^*) \leq 0$ for $y \in \mathcal{K}_d$. □

(f) The **generalization** of the basic alternative Theorem 4 and of theorem 5 by incorporating affin linear side conditions $h(x) = 0$ makes some difficulties because the trivial cone $\{0\}$ has an empty interior. [Kirsch et al.] develop an own theory and abandon the requirement for *closed* separating hyperplane. F.JOHN generalizes the above result by studying the "primal functional" which is also of interest in duality theory. We follow here the latter way.

Recall that $\mathcal{R} := \mathcal{P} + \mathcal{Q} := \{r = p + q,\ p \in \mathcal{P}, q \in \mathcal{Q}\}$ for some subsets of a vector space. If $\mathcal{P}$ and $\mathcal{Q}$ are convex then also $\mathcal{P} + \mathcal{Q}$ is convex and

$$\mathrm{int}(\mathcal{P}) \neq \emptyset \Longrightarrow \mathrm{int}(\mathcal{R}) \neq \emptyset$$

as shown in the proof of Theorem 4. Define the sets

$$\begin{aligned} \Gamma &= g(\mathcal{C}) + \mathcal{K} = \{g(x) + k,\ x \in \mathcal{C},\ k \in \mathcal{K}\}, \\ \mathcal{A} &= (g, f)(\mathcal{C}) + (\mathcal{K} \times \mathbb{R}_{\geq 0}) = \{(g(x) + k, f(x) + \alpha),\ x \in \mathcal{C},\ k \in \mathcal{K},\ \alpha \geq 0\}, \end{aligned}$$

the primal functional

$$\begin{aligned} \omega : \Gamma \ni u \mapsto \omega(u) &:= \inf\{w,\ (u, w) \in \mathcal{A}\} \\ &= \inf\{f(x),\ x \in \mathcal{C},\ \exists\, k \in \mathcal{K} : g(x) + k = u\}, \end{aligned}$$

and the set

$$\mathcal{B} = \{(u, w) \in Y \times \mathbb{R},\ u \leq 0,\ w \leq \omega(0)\}.$$

Then, in particular, $(g(x), f(x)) \in \mathcal{A}$ for all $x \in \mathcal{C}$, and ω is the boundary of $\mathcal{A}$. We assemble some properties in the following auxiliary result:

Lemma 4 *Let the minimum problem* $\{f(x),\ x \in \mathcal{C},\ g(x) \leq 0\} = \min!$, *be convex and let* $\omega(0)$ *exist finitely. Then*
(i) Γ *is convex.*
(ii) ω *is convex and weakly monotone decreasing.*
(iii) $\mathcal{A}$ *and* $\mathcal{B}$ *are convex.*
(iv) $\mathrm{int}(\mathcal{A}) \cap \mathcal{B} = \emptyset$.
(v) $\mathcal{A} \cap \mathrm{int}(\mathcal{B}) = \emptyset$.

Proof [Luenber69], §8.3; [Kosmol] §9.3. All assertions are rather obvious besides the first assertion of (ii). For this proof let $u, v \in \mathcal{Y}$ and $0 < \lambda < 1$. Define

$$\begin{aligned} \mathcal{P} &:= \{x \in \mathcal{C},\ \exists\, k \in \mathcal{K} : g(x) \leq \lambda u + (1 - \lambda)v\}, \\ \mathcal{Q} &:= \{x \in \mathcal{C},\ \exists\, x_1, x_2 \in \mathcal{C},\ \exists\, k_1, k_2 \in \mathcal{K} : \\ &\qquad g(x_1) + k_1 = u,\ g(x_2) + k_2 = v,\ x = \lambda x_1 + (1 - \lambda)x_2\}. \end{aligned}$$

If $x \in \mathcal{Q}$ then there exist $x_1, x_2 \in \mathcal{C}$ such that

$$g(x) = g(\lambda x_1 + (1-\lambda)x_2) \le \lambda g(x_1) + (1-\lambda)g(x_2) \le \lambda u + (1-\lambda)v$$

therefore $x \in \mathcal{Q} \Longrightarrow x \in \mathcal{P}$ and thus $\mathcal{Q} \subset \mathcal{P}$. Now

$$\omega(\lambda u + (1-\lambda)v) = \inf\{f(x),\ x \in \mathcal{P}\} \le \inf\{f(x),\ x \in \mathcal{Q}\}.$$

But f is convex hence

$$f(x) = f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2)$$

and therefore

$$\begin{aligned} &\inf\{f(x),\ x \in \mathcal{Q}\} \\ &\le \inf\{\lambda f(x_1) + (1-\lambda)f(x_2),\ x_1 \in \mathcal{C}, g(x_1) \le u,\ x_2 \in \mathcal{C},\ g(x_2) \le v\} \\ &= \lambda \inf\{f(x_1),\ x_1 \in \mathcal{C},\ g(x_1) \le u\} + (1-\lambda)\inf\{f(x_2),\ x_2 \in \mathcal{C},\ g(x_2) \le v\} \\ &= \lambda\omega(u) + (1-\lambda)\omega(v).\ \Box \end{aligned}$$

By and large, the following result is the same as Theorem 5 but allows a generalization to additional affine linear side conditions.

Theorem 6 (F. JOHN) *Let the minimum problem (1) be convex, let $h \equiv 0$, and let $\omega(0)$ exist finitely.*
(i) If $\operatorname{int}(\mathcal{A}) \ne \emptyset$ then there exists a pair $(0,0) \ne (\varrho^*, y^*) \in \mathbb{R}_{\ge 0} \times \mathcal{K}_d$ such that

$$\varrho^*\omega(0) = \inf\{\varrho^* f(x) + y^* \circ g(x),\ x \in \mathcal{C}\}. \tag{15}$$

(ii) If $\operatorname{int}(\mathcal{K}) \ne \emptyset$ and the SLATER condition is satisfied then (15) holds with $\varrho^* = 1$ und $0 \ne y^* \in \mathcal{K}_d$.
(iii) If moreover x^* is a solution of the minimum problem then x^* minimizes also (15) and

$$y^* \circ g(x^*) = 0.$$

Proof. [Luenberger69], § 8.3; [Kosmol], § 9.3.
(i) The convex sets $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumption of EIDELHEIT's separation theorem thus there exists a closed separating hyperplane. This is equivalent to the existence of a pair $(0,0) \ne (\varrho^*, y^*) \in (\mathbb{R} \times Y)_d = \mathbb{R} \times Y_d$ such that

$$\forall\ (u_1, w_1) \in \mathcal{B},\ \forall\ (u_2, w_2) \in \mathcal{A}:\ \ \varrho^* w_1 + y^* \circ u_1 \le \varrho^* w_2 + y^* \circ u_2. \tag{16}$$

By the shape of $\mathcal{B}$ we obtain $\varrho^* \ge 0$ and $y^* \ge 0$: $\varrho^* < 0$ contradicts (16) because w_2 may be arbitrarily large. Also, if $y^* \circ u_1 > 0$ for some $u_1 < 0$ then

$$y^* \circ n\, u_1 = n\, y^* \circ u_1,\ n \in \mathbb{N}$$

may be arbitrarily large, contradicting (16), too. So, $y^* \circ u \le 0$ for all $u < 0$ yielding $y \in \mathcal{K}_d$ by the cone inclusion theorem. Because $(0, \omega(0)) \in \mathcal{B}$ and $(g(x), f(x)) \in \mathcal{A}$ for all $x \in \mathcal{C}$ we have

$$\varrho^*\omega(0) \le \inf\{\varrho^* f(x) + y^* \circ g(x),\ x \in \mathcal{C}\}. \tag{17}$$

If $\{x_n\}$ is a sequence of feasible points with $\lim f(x_n) = \omega(0)$ then

$$\inf\{\varrho^* f(x) + y^* \circ g(x),\ x \in \mathcal{C}\} \le \lim \varrho^* f(x_n) = \varrho^*\omega(0).$$

because $y^* \circ g(x_n) \le 0$. This proves (i).
(ii) If $\mathrm{int}(\mathcal{K}) \neq \emptyset$ then $\mathrm{int}(\mathcal{A}) \neq \emptyset$ and the assumption of EIDELHEIT's Separation Theorem is fulfilled and assertion (i) holds. But the SLATER condition yields the existence of some $x_0 \in \mathcal{C}$ such that $g(x_0) < 0$. Then $y \circ g(x_0) < 0$ for all $0 \neq y \in \mathcal{K}_d$ by the cone corollary. But $\varrho^* = 0$ implies $y^* \neq 0$ and then in (16)

$$0 = y^* \circ 0 \le y^* \circ g(x_0),$$

contradicting $y^* \circ g(x_0) < 0$. Therefore we now have $\varrho^* > 0$ and assertion (ii) follows after division by ϱ^*.
(iii) If x^* solves the minimum problem then we obtain from (17) using $\varrho^* = 1$ and $y^* \ge 0$

$$\omega(0) \le f(x^*) + y^* \circ g(x^*) \le f(x^*) = \omega(0)$$

yielding both parts of assertion (iii). □
The proof of (ii) shows that the SLATER condition may be replaced by the somewhat weaker condition

$$\exists\, x \in \mathcal{C} \ \ \forall\, 0 \neq y \in \mathcal{K}_d : y \circ g(x) < 0.$$

However, the following somewhat stronger result holds for $\mathcal{A}$:

Lemma 5 *Let $\mathcal{D} \subset \mathcal{C}$, $\mathrm{int}(g(\mathcal{D})) \neq \emptyset$ or $\mathrm{int}(\mathcal{K}) \neq \emptyset$ and let $f(\mathcal{D}) \subset \mathbb{R}$ be bounded. Then $\mathrm{int}(\mathcal{A}) \neq \emptyset$.*

Proof. [Kosmol], § 9.3. The interval $(\sup f(\mathcal{D}), \infty) \subset \mathbb{R}$ is open and nonempty by assumption. But $\mathrm{int}(g(\mathcal{D}) + \mathcal{K}) \neq \emptyset$ by assumption hence also the set $\mathcal{G} := (g(\mathcal{D}) + \mathcal{K}) \times \{(\sup f(D), \infty)\} \subset Y \times \mathbb{R}$ has a nonempty interior. But $\mathcal{G} \subset \mathcal{A}$ holds because

$$(g(x) + k, \beta) = (g(x) + k, f(x) + (\beta - f(x)) \in \mathcal{A}$$

for every $(g(x) + k, \beta) \in \mathcal{G}$ observing $\beta - f(x) \ge 0$. □
Theorem 6 and Lemma 5 together yield the following useful result:

Corollary 3 *[Kosmol], § 9.3, Folgerung. Let the minimum problem (1) be convex, let $h \equiv 0$, and let $\omega(0)$ exist finitely.*
(1°) Let there exist a set $\mathcal{D} \subset \mathcal{C}$ such that
(1°)(i) f is bounded from above on $\mathcal{D}$,
(1°)(ii) $\mathrm{int}(g(\mathcal{D})) \neq \emptyset$ or $\mathrm{int}(\mathcal{K}) \neq \emptyset$.
Then there exists a pair $(0,0) \neq (\varrho^*, y^*) \in \mathbb{R}_{\ge 0} \times \mathcal{K}_d$ such that

$$\varrho^* \omega(0) = \inf\{\varrho^* f(x) + y^* \circ g(x),\ x \in \mathcal{C}\}. \tag{18}$$

(2°) If moreover

$$\exists\, x \in \mathcal{C},\ \forall\, 0 \neq y \in \mathcal{K}_d : y \circ g(x) < 0.$$

then (18) holds with $\varrho^* = 1$ und $0 \neq y^* \in \mathcal{K}'$.
(c) If moreover x^* is a solution of the minimum problem then x^* minimizes also (18) and

$$y^* \circ g(x^*) = 0.$$

(g) Linear Constraints

Lemma 6 *Let* $\mathcal{X}$, $\mathcal{Z}$ *be* BANACH *spaces. Let* $f : \mathcal{X} \to \mathbb{R}$ *be convex and continuous,* $0 \neq h : \mathcal{X} \to \mathcal{Z}$ *linear and continuous,* $h(\mathcal{X})$ *closed and* $b \in h(\mathcal{X})$*. Further, let*

$$\omega := \inf\{f(x),\ x \in \mathcal{X},\ h(x) - b = 0\} \tag{19}$$

exist finitely.
(1°) There exists a $z^* \in \mathcal{Z}_d$ *such that*

$$\omega = \inf\{f(x) + z^* \circ (h(x) - b),\ x \in \mathcal{X}\}\,. \tag{20}$$

(2°) If the infimum in (19) is attained at a point x^* *with* $h(x^*) - b = 0$ *then the infimum of (20) is also attained in* x^* *and moreover*

$$z^* \circ (h(x^*) - b) = 0\,. \tag{21}$$

(3°) If conversely the infimum of (20) is attained at a point x^* *such that* $h(x^*) = b$ *then also the infimum of (19) is attained at* x^* *.*

Proof. [Kosmol], §9.4. By assumption $h(\mathcal{X})$ is closed hence a BANACH space. Let $\mathcal{D} \subset \mathcal{X}$ be an open bounded ball. Then f is bounded on $\mathcal{D}$ because f is continuous. $h(\mathcal{D}) \subset h(\mathcal{X})$ is an open subset by the Open-Mapping-Theorem being bounded also. Because $h(\mathcal{X})$ is a vector space and $b \in h(\mathcal{X})$, also $g(\mathcal{D}) = h(\mathcal{D}) - b \subset h(\mathcal{X})$ and $\mathrm{int}(g(\mathcal{D})) = g(\mathcal{D}) \neq \emptyset$ where $\mathrm{int}(g(\mathcal{D}))$ is now the interior of $g(\mathcal{D}))$ relative $h(\mathcal{X})$. Thus the assumptions of Corollary 3 are fulfilled for the BANACH space $h(\mathcal{X})$ instead of $\mathcal{Y}$ and consequently there exists a pair $(0,0) \neq (\varrho^*, z^*)$ with $z^* \in h(\mathcal{X})_d$ and

$$\varrho^* \omega = \inf\{\varrho^* f(x) + z^* \circ (h(x) - b),\ x \in \mathcal{X}\}. \tag{22}$$

If $\varrho^* = 0$ then $z^* \neq 0$ and then from (22)

$$\forall\, x \in X : z^* \circ (h(x) - b) \geq 0$$

in contradiction to the fact that $h \neq 0$ and $h(\mathcal{X})$ is a vector space. Therefore $\varrho^* \neq 0$ and we may choose $\varrho^* = 1$. By the theorem of HAHN-BANACH z^* can be continued to Z_d retaining its properties.
Part (3°) follows in the same way as in Theorem 6. □
We also may write Lemma 6 in the following form using an affin linear mapping h:

Corollary 4 *Let* $\mathcal{X}$, $\mathcal{Z}$ *be* BANACH *spaces,* $f : \mathcal{X} \to \mathbb{R}$ *convex and continuous,* $0 \neq h : \mathcal{X} \to \mathcal{Z}$ *affin linear and continuous,* $h(\mathcal{X})$ *closed and* $0 \in h(\mathcal{X})$*. Further, let*

$$\omega := \inf\{f(x),\ x \in \mathcal{X},\ h(x) = 0\}$$

exist finitely. Then there exists a $z^* \in \mathcal{Z}_d$ *such that*

$$\omega = \inf\{f(x) + z^* \circ h(x),\ x \in \mathcal{X}\}.$$

If moreover

$$x^* = \arg\min\{f(x),\ x \in \mathcal{X},\ h(x) = 0\}$$

then

$$x^* = \arg\min\{f(x) + z^* \circ h(x),\ x \in \mathcal{X}\}$$

and vice versa (and $z^* \circ h(x^*) = 0$ *in a trivial way).*

Definition 4 *Let $\mathcal{C}, \mathcal{D} \subset \mathcal{X}$ arbitrary sets.*
(1°) *Let $x \in \mathcal{C} \cap \mathcal{D}$ then x is called* interior point of $\mathcal{C}$ relative to $\mathcal{D}$ *if there is a neighborhood of x in $\mathcal{D}$ being also neighborhood of x in $\mathcal{C}$:*

$$\exists\, \varepsilon > 0,\ \forall\, u \in \mathcal{D} : \|u - x\| \le \varepsilon \Longrightarrow u \in \mathcal{C}.$$

(2°) aff$(\mathcal{C})$ *is the smallest affine subspace of $\mathcal{X}$ containing $\mathcal{C}$.*

Lemma 7 *Let the minimum problem (1),*

$$\{f(x)\,;\ x \in \mathcal{C}\,,\ h(x) = 0\} = \min!,$$

be convex (h affine linear). Suppose that,
(i) $\inf f(\mathcal{C})$ *exists finitely,*
(ii) $h : \mathcal{C} \to \mathbb{R}_m$ *finite-dimensional,*
(iii) $0 \in h(\mathcal{C})$.
Then there exists a pair $(0,0) \neq (\varrho^*, z^*) \in \mathbb{R} \times \mathbb{R}_m$ *such that*

$$\varrho^* \inf\{f(x),\ x \in \mathcal{C},\ h(x) = 0\} = \inf\{\varrho^* f(x) + z^*\, h(x),\ x \in \mathcal{C}\}.$$

If moreover $0 \in h(\mathcal{C})$ is interior point relative to aff$(h(\mathcal{C}))$ *then $\varrho^* \neq 0$.*

Proof. [Kosmol] § 9.5. Let $\mathcal{U} \subset \mathcal{Z}$ be the smallest affin linear subspace containing $h(\mathcal{C})$ and $\mathcal{K} = \{0\}$. Because $0 \in h(\mathcal{C})$, $\mathcal{U}$ is a proper subspace with a basis $z_1, \ldots, z_k$. Writing $z_0 = 0$ there exist $x_i \in \mathcal{C},\ i = 0, \ldots k$, with $z_i = h(x_i)$. For the convex hull $\mathcal{D} \subset \mathcal{C}$ of these points we obtain:
(a) f is bounded on $\mathcal{D}$: For $\alpha_0, \ldots, \alpha_k \in [0,1]$ with $\alpha_0 + \ldots + \alpha_k = 1$ we have

$$f\Big(\sum_{i=0}^{k} \alpha_i x_i\Big) \le \sum_{i=0}^{k} \alpha_i f(x_i) \le \mathrm{Max}\{|f(x_i)|,\ i = 0, \ldots, k\}.$$

(b) $\mathrm{int}\, h(\mathcal{D}) \neq \emptyset$: $h(\mathcal{D})$ is the convex hull of $0, z_1, \ldots, z_k$ because h is affin linear and $0 \in h(\mathcal{C})$. As $z_1, \ldots, z_k$ is a basis of $\mathcal{U}$, $\mathrm{int}\, h(\mathcal{D})$ is not empty in the vector space $\mathcal{U}$. Accordingly, the assumption of Corollary 3 is fulfilled for $\mathcal{U}$ instead of Y and there exists a pair $(0,0) \neq (\varrho^*, z^*) \in \mathbb{R}_{\ge 0} \times \mathcal{U}_d \subset \mathbb{R}^{m+1}$ such that

$$\varrho^* \inf\{f(x),\ x \in \mathcal{C},\ h(x) = 0\} = \inf\{\varrho^* f(x) + z^{*T} h(x),\ x \in \mathcal{C}\}.$$

If $\varrho^* = 0$ then $z^* \neq 0$ and this equation yields

$$\forall\, x \in \mathcal{C} : 0 \le z^{*T} h(x).$$

This inequality however contradicts $0 \neq z^* \in \mathcal{U}_d$ if $0 \in h(\mathcal{C})$ is an interior point relative to the linear span $\mathcal{U}$ of $h(\mathcal{C})$. □

(h) Mixed Constraints

Theorem 7 *(Existence) Let the minimum problem (1),*

$$\{f(x)\,;\ x \in \mathcal{C}\,,\ g(x) \le 0\,,\ h(x) = 0\} = \min!,$$

be convex (h affine linear), let $\mathcal{G} = \{x \in \mathcal{C}\,,\ g(x) \le 0\}$. Suppose that
(1°) $\mathrm{int}(\mathcal{K}) \neq \emptyset$,

(2°) $h : \mathcal{C} \to Z = \mathbb{R}^m$,
(3°)*(i)* $\exists\, x \in \mathcal{C} : g(x) < 0$ *and* $0 \in h(\mathcal{C})$ *is interior point relative to* $\operatorname{aff}(h(\mathcal{G}))$
or
(3°)*(ii)* $0 \in h(\mathcal{C})$ *is interior point relative to* $\operatorname{aff}(h(\mathcal{C}))$ *and* $\exists\, x \in \mathcal{C} : g(x) \le 0\,,\ h(x) = 0.$
(4°)

$$\omega := \inf\{f(x),\ x \in \mathcal{C},\ g(x) \le 0,\ h(x) = 0\} \tag{23}$$

exists finitely.
Then there exists a pair $(y^*, z^*) \in \mathcal{K}_d \times \mathcal{Z}_d$ *with* $y^* \ne 0$ *such that*

$$\omega = \inf\{f(x) + y^* \circ g(x) + z^* \circ h(x),\ x \in \mathcal{C}\}. \tag{24}$$

If the infimum of (23) is attained at a feasible x^* *then* x^* *minimizes also (24) and* $y^* \circ g(x^*) = 0$.

Proof. for (3°)(ii). Let $\widetilde{\mathcal{C}} = \{x \in \mathcal{C},\ h(x) = 0\}$. By Theorem 6 there exists a $0 \ne y^* \in \mathcal{K}_d$ such that

$$\begin{aligned} \omega &= \inf\{f(x),\ x \in \widetilde{\mathcal{C}},\ g(x) \le 0\} \\ &= \inf\{f(x) + y^* \circ g(x),\ x \in \mathcal{C},\ h(x) = 0\}. \end{aligned}$$

Because of the regularity assumption (3°)(ii)) there exists a $z^* \in \mathbb{R}^m$ by Lemma 7 such that

$$\omega = \inf\{f(x) + y^* \circ g(x) + z^{*T} h(x),\ x \in \mathcal{C}\}.$$

For (3°)(i). Let $\widetilde{\mathcal{C}} = \{x \in \mathcal{C},\ g(x) \le 0\}$. By Lemma 7 there exists a $z^* \in \mathbb{R}_m$ such that

$$\begin{aligned} \omega &= \inf\{f(x),\ x \in \widetilde{\mathcal{C}},\ h(x) = 0\} \\ &= \inf\{f(x) + z^*\, h(x),\ x \in \widetilde{\mathcal{C}}\}. \end{aligned}$$

Farkas' Theorem, applied to the convex function

$$f_1 : \mathcal{C} \to \mathbb{R}\,,\ f_(x) = f(x) + z^* \circ h(x)\,,$$

supplies a $y^* \in \mathcal{K}_d$ such that

$$\omega = \inf\{f(x) + z^*\, h(x)\,,\ x \in \widetilde{\mathcal{C}}\} = \inf\{f(x) + z^*\, h(x) + y^* \circ g(x)\,,\ x \in \mathcal{C}\}\,.$$

The last assertion follows in the same way as part (iii) of Theorem 6.□

Theorem 8 *((SPP) necessary in convex problems) Adopt the assumptions of Theorem 7. Then there exists a pair* $(0,0) \ne (y^*, z^*) \in K_d \times Z_d$ *such that*

$$\forall\ (x, y, z) \in \mathcal{C} \times K_d \times Z_d : L(x^*, y, z) \le L(x^*, y^*, z^*) \le L(x, y^*, z^*)$$

and $y^* \circ g(x^*) = 0$.

Because of Theorem 7, only the left inequality has to be verified. But it is equivalent to $y \circ g(x^*) \le y^* \circ g(x^*) = 0$ and is thus true because $y \ge 0$ and $g(x^*) \le 0$. □
Consider now the augmented minimum problem

$$\{f(x),\ x \in \mathcal{X},\ g(x) \le 0,\ h(x) = 0,\ r(x) = 0\} = \min\,.$$

Theorem 9 *Suppose that*
(i) $\mathcal{X}, \mathcal{Z}$ *are* BANACH *spaces,* $\mathcal{Y}$ *is normed with order cone* $\mathcal{K} \subset \mathcal{Y}$ *,*
(ii) $f : \mathcal{X} \to \mathbb{R}$ *is convex,* $g : \mathcal{X} \to \mathcal{Y}$ *is* $\mathcal{K}$*-convex,*
(iii) $h : \mathcal{X} \to \mathbb{R}^m$ *is affine linear,*
(iv) $r : \mathcal{X} \to \mathcal{Z}$ *is affine linear and continuous,*
(v) $\mathrm{int}(\mathcal{K}) \neq \emptyset$*,*
(vi) $\exists\, x \in \mathcal{X} : g(x) < 0,\ h(x) = 0,\ r(x) = 0,$
(vii) $r(\mathcal{X})$ *closed,*
(viii)

$$\omega := \inf\{f(x),\ x \in \mathcal{X},\ g(x) \leq 0,\ h(x) = 0,\ r(x) = 0\} \tag{25}$$

exists finitely.
Then there exists a triple $(y^*, z^*, w^*) \in \mathcal{K}_d \times \mathbb{R}_m \times \mathcal{Z}_d$ *with* $y^* \neq 0$ *such that*

$$\omega = \inf\{f(x) + y^* \circ g(x) + z^* \circ h(x) + w^* \circ r(x),\ x \in \mathcal{X}\} \tag{26}$$

If moreover x^* *is a solution of (25) then* x^* *also minimizes (26) and* $y^* \circ g(x^*) = 0$*.*

Proof. Let now $\widetilde{\mathcal{C}} = \{x \in \mathcal{X},\ r(x) = 0\}$ then we obtain by Theorem 7

$$\omega = \inf\{f(x) + y^* \circ g(x) + w^* \circ h(x),\ x \in \widetilde{\mathcal{C}}\}$$

The assertion then follows by Corollary 4 for $F = f + y^* \circ g + w^* h$ instead of f . The last part follows in the same way as part (c) in Theorem 6. □

Theorem 10 *((SPP) necessary in convex problems) Adopt the assumptions of Theorem 9 and let* $h = 0$ *Then there exists a pair* $(0, 0) \neq (y^*, z^*) \in K_d \times \mathcal{Z}_d$ *such that*

$$\forall\ (x, y, z) \in \mathcal{C} \times \mathcal{K}_d \times \mathcal{Z}_d : L(x^*, y, z) \leq L(x^*, y^*, z^*) \leq L(x, y^*, z^*)$$

and $y^* \circ g(x^*) = 0$*.*

Proof. The assertion follows from Theorem 9 by setting $h = 0$. The left inequality is proved as in Theorem 8.
The results concerning affine linear and mixed constraints are due to [Kosmol].