## Supplements 3 to Chapter III

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### Linear-Quadratic Programming

$$\boxed{\begin{array}{rcl} f(x) & = & \frac{1}{2}x^TAx - a^Tx = \min! \\ h(x) & = & Bx + b = 0 \end{array}} \,. \tag{1}$$

$$\begin{bmatrix} H & [B^{\mathcal{A}}]^T \\ B^{\mathcal{A}} & 0 \end{bmatrix}\begin{bmatrix} d \\ y^{\mathcal{A}} \end{bmatrix} = \begin{bmatrix} \nabla f(x)^T \\ 0 \end{bmatrix}. \tag{2}$$

**Lemma 1** *The direction $d$ in (2) is non-zero after inactivation of an active constraint hence actually $f(x - \sigma d) < f(x)$ .*

More generally formulated:

**Theorem 1** *After a step of incativation the nes search direction $d$ with $\mathcal{A}(x)\backslash\{i\}$ instead $\mathcal{A}(x)$ satisfies*
(°)

$$\nabla f(x)d \geq \alpha\|d\|^2, \quad (A \geq \alpha I)$$

(2°)

$$b^id < 0, (b^ix - \beta^i > 0, b^i(x - \sigma d) - \beta^i > 0).$$

*Proof.* (a) Without loss of generality, let $|\mathcal{A}(x)| = r + 1, r \in \mathbb{N}_0$ and $i = r + 1$ as well as $d = 0$ and $c = b^i$ (row). Then

$$\begin{bmatrix} A & [B^{\mathcal{A}}]^T & c^T \\ B^{\mathcal{A}} & O & O \\ c & 0 & 0 \end{bmatrix}\begin{bmatrix} d \\ w \\ \zeta \end{bmatrix} = \begin{bmatrix} \nabla f(x)^T \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

where $d = 0$, $\zeta < 0$. *After* inactivation of row $g^i(x) = 0$ we have

$$\begin{bmatrix} A & [B^{\mathcal{A}}]^T \\ B^{\mathcal{A}} & O \end{bmatrix}\begin{bmatrix} \widetilde{d} \\ \widetilde{w} \end{bmatrix} = \begin{bmatrix} \nabla f(x)^T \\ 0 \end{bmatrix} \tag{4}$$

**(a)** Let $d = 0$ and $\widetilde{d} = 0$ then

$$\nabla f(x) \in \text{Range}[[B^{\mathcal{A}}]^T, c^t]] \text{ und } \nabla f(x) \in \text{Range}[B^{\mathcal{A}}]^T.$$

This is a contradiction since $[B^{\mathcal{A}}, c]$ rank-maximal and $\zeta \neq 0$.
**(b)** From (4) we obtain

$$\alpha\widetilde{d}^T\widetilde{d} \leq \widetilde{d}^TA\widetilde{d} + \widetilde{d}^TB^{\mathcal{A}}\widetilde{w} = \widetilde{d}^T\nabla f(x)^T$$

By (4) we have $\widetilde{d}^TB^{\mathcal{A}}\widetilde{w} = 0$ hence $\nabla f(x)\widetilde{d} \geq \alpha|\widetilde{d}|^2$. From (3) we obtain, because $d = 0$ ,

$$[B^{\mathcal{A}}]^Tw + c^T\zeta = \nabla f(x)^T$$
$$\widetilde{d}^T[B^{\mathcal{A}}]^Tw + \widetilde{d}^Tc^T\zeta = \widetilde{d}^T\nabla f(x)^T > 0$$

by the above equation therefore $\widetilde{d}^Tc^T = b^i\widetilde{d} < 0$.

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**(f1)** In In Algorithm § 3.5 **(e)** the penalty parameters become unnecessarily large near the solution with bad effect on numerical stability. Therefore the following modification of (2°) is proposed for computation of these weights where however the initial vector $x_0$ is involved:

Additional initial parameters $0 < \eta \ll 1\,,\ c = 1\,.$ The numerator $c$ is always enhanced by one if the penalties are lowered.
(2.1°) Set $\gamma = \widetilde{y} + \varepsilon \in \mathbb{R}^m\,,\ \delta = |\widetilde{z}| + \varepsilon \in \mathbb{R}^p\,,$
set flag $= 0$ if $\gamma \geq y$ and $\delta \geq z$, flag $= 1$ else.
(2.2°) If flag $= 1$ and $P(x_0, \gamma, \delta) - P(x, \gamma, \delta) \geq c\eta$, then set $c := c + 1\,,\ y = \gamma\,,\ z = \delta$
otherwise *replace* $y_i$, $z_i$ for all $i$ by

$$\begin{array}{lll} y_i & := & \widetilde{y}_i + 2\varepsilon \quad \text{if} \quad \widetilde{y}_i + \varepsilon \geq y_i \\ z_i & := & |\widetilde{z}_i| + 2\varepsilon \quad \text{if} \quad |\widetilde{z}_i| + \varepsilon \geq z_i \end{array}$$

By this way the penalties $y$ and $z$ may be reduced again and altogether they are adapted more properly to geometric constellation of the individual step of iteration.