Supplements 1 to Chapter III

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To Section 3.2

$$\boxed{\{f(x)\,;\; g(x)\le 0\,,\; h(x)=0\} = \min!} \tag{1}$$

Theorem 1 *(MR sufficient in convex problems) Let the minimum problem (1) be convex (hence h affine linear) and differentiable, then every* KUHN-TUCKER *point is a global minimum point of f in $\mathcal{S}$ and thus a solution of the minimum problem.*

Proof We carry the proof out separately for equalities h and inequalities g altough both parts are rather similar. Of Course both results may be combined with each other.
(1°) Since f convex and h affine linear, by Lemma 1.28

$$\begin{aligned} f(x) &\ge f(x^*) + \nabla f(x^*)(x-x^*)\,, \\ h(x) &= h(x^*) + \nabla h(x^*)(x-x^*)\,. \end{aligned}$$

Accordingly, for all x satisfying $h(x)=0$ and all z^*

$$\begin{aligned} f(x) &= f(x) + z^*h(x) \\ &\ge f(x^*) + \nabla f(x^*)(x-x^*) + z^*h(x^*) + z^*\nabla h(x^*)(x-x^*) \\ &= f(x^*) + [\nabla f(x^*) + z^*\nabla h(x^*)](x-x^*) \overset{\text{MR}}{=} f(x^*). \end{aligned}$$

Only in the last equation, z^* must be specified by the MR.
(2°) Since f and g convex, for all x satisfying $g(x)\le 0$ and for $y^*\ge 0$ by Lemma 1.28

$$\begin{aligned} f(x) &\ge f(x) + y^*g(x) \\ &\ge f(x^*) + \nabla f(x^*)(x-x^*) + y^*g(x^*) + y^*\nabla g(x^*)(x-x^*) \\ &= f(x^*) + [\nabla f(x^*) + y^*\nabla g(x^*)](x-x^*) \overset{\text{MR}}{=} f(x^*). \end{aligned}$$

□

To Section 3.3

Detailed treatment in `KAPITEL03\LOP`.

$$\boxed{\min\{a\,x\,;\; Bx\le c\}\,,\quad a\in\mathbb{R}_n\,,\quad c\in\mathbb{R}^m} \tag{2}$$

$$\begin{aligned} \mathcal{A}^*(x) &:= \{k\in\{1,\ldots,m\}\,,\; b^kx=\gamma^k\}\,, \\ \mathcal{A}(x) &:= \{k\in\mathcal{A}^*(x)\,,\; b^k \text{ linearly independent}\}, \\ \mathcal{N}(x) &:= \{1,\ldots,m\}\backslash\mathcal{A}(x)\,. \end{aligned}$$

$\mathcal{A}^*(x)$ denotes the *(index-)set of active side conditions* in x. The set $\mathcal{N}(x)$ contains the indices of all inactive side conditions in x if and only if $\mathcal{A}(x)$ coincides with $\mathcal{A}^*(x)$. In the other case, neither $\mathcal{A}(x)$ nor $\mathcal{N}(x)$ are determined uniquely.

Lemma 1 *$x\in\mathcal{S}$ is extreme point if and only if $|\mathcal{A}(x)| = n$.*

Proof. (a) Let $|\mathcal{A}(x)| = n$ then the system of equations

$$b^j x = \gamma^j,\ j \in \mathcal{A}(x),$$

has a unique solution x. Let $x + \alpha\, v \in \Omega$, i.e. $B(x + \alpha\, v) \le c$ then $\alpha\, b^j v \le 0$ must be fulfilled for all $j \in \mathcal{A}(x)$ therefore $\alpha \ge 0$ or $\alpha \le 0$ must be valid but then x cannot be a genuine convex combination hence x is a corner point.
(b) Let $|\mathcal{A}(x)| < n$ and, without loss of generality, let $b^n = \sum_{i=1}^{n-1} s_j b^j$. There exists a $0 \ne v \in \mathbb{R}^n$ such that $b^j v = 0,\ j = 1, \dots, n-1$, hence

$$\begin{aligned} b^j(x + \tau v) &= \gamma^j, \quad j = 1, \dots, n-1, \quad \tau \in \mathbb{R}, \\ b^n(x + \tau v) &= (\textstyle\sum_{j=1}^{n-1} s_j b^j)(x + \tau v) = \sum_{j=1}^{n-1} s_j b^j x = b^n x = \gamma^n. \end{aligned}$$

Therefore $x + \tau v \in \Omega$ for all sufficiently small $\tau \in \mathbb{R}$ (because of the inactive restrictions), therefore x cannot be corner point. □

Theorem 2 *(Existence) Let the feasible domain $\mathcal{S}$ be non-empty and let the objective function $x \mapsto ax$ be bounded on $\mathcal{S}$. Then the problem (2) has a solution being extreme point of $\mathcal{S}$.*

Proof. The problem has a solution in $\mathcal{S}$ by assumption. We show: If $u \in \mathcal{S}$, there exists a corner $v \in \mathcal{S}$ mit $av \ge au$. Without loss of generality let

$$\begin{aligned} & b^j u = \gamma^j, \quad j = 1, \dots, k, \\ & b^j u < \gamma^j, \quad j = k+1, \dots, m, \\ & b^j \text{ linear unabh"angig}, \quad j = 1, \dots, k. \end{aligned}$$

The other active side conditions may be omitted entirely in this consideration. Let $k = n$ then $v = u$ is corner point by Lemma 3.1. If $k < n$, there exists a search direction s such that

$$b^j s = 0, \quad j = 1, \dots, k,$$

(s arbitrary if $k = 0$). Let without loss of generality $as \ge 0$ else replace s by $-s$. Then

$$a(u + \sigma s) \ge au \quad \forall\, \sigma > 0 \tag{3}$$

(for all $\sigma \in \mathbb{R}$ if $as = 0$). Choose σ maximal such that

$$b^j(u + \sigma s) \le \gamma^j, \quad j = k+1, \dots, m,$$

then $v = u + \sigma s$ is feasible and

$$\sigma b^j s \le \gamma^j - b^j u\ (> 0), \quad j = k+1, \dots, m, \tag{4}$$

since u has been feasible.
Case 1. There exists a $j \in \{k+1, \dots, m\}$ such that $b^j s > 0$. Then

$$\begin{aligned} \nu &= \operatorname{Arg\,Min}\{f(j) := \frac{\gamma^j - b^j u}{b^j s},\ b^j s > 0,\ j = k+1, \dots, m\} \\ \sigma &:= f(\nu) > 0. \end{aligned}$$

Without loss of generality, let $\nu = k+1$, then

$$\begin{aligned} b^{k+1}(u + f(\nu)s) &= b^{k+1}u + \frac{\gamma^{k+1} - b^{k+1}u}{b^{k+1}s} b^{k+1}s \\ &= b^{k+1}u + \gamma^{k+1} - b^{k+1}u = \gamma^{k+1}. \end{aligned}$$

Furthermore,

$$\begin{aligned} b^j(u+f(\nu)s) &< \gamma^j, \quad j=k+2,\dots,m,\\ b^j(u+f(\nu)s) &= \gamma^j, \quad j=1,\dots,k. \end{aligned}$$

Thus $k+1$ conditions are now active in $v=u+\sigma s\in\Omega$ and $av\geq au$. Besides, b^{k+1} is linearly independent of $b^1,\dots,b^k$ because

$$b^{k+1}s>0, \quad b^js=0,\ j=1,\dots,k,$$

hence b^{k+1} is not contained in the hyperplane $s^Tx=0$, but $b^j,\ j=1,\dots,k$ do so.

Case 2. Let $b^js\leq 0$ for all $j=k+1,\dots,m$.

By assumption we have $b^js=0,\ j=1,\dots,k<n$. Hence there exists a $j\in\{k+1,\dots,m\}$ such that $b^js<0$. Else $b^js=0,\ j=1,\dots,m\geq n$, i.e. all b^j are contained in the hyperplane $s^Tx=0$ in contradiction to the assumption $\mathrm{Rang}(B)=n$. This second case appears only if the set Ω is unbounded. Then

$$\begin{aligned} b^j(u+\sigma s) &= \gamma^j, \quad \forall\,\sigma>0,\ j=1,\dots,k,\\ b^j(u+\sigma s) &\leq \gamma^j, \quad \forall\,\sigma>0,\ j=k+1,\dots,m. \end{aligned}$$

Therefore $u+\sigma s\in\mathcal{S}$ for all $\sigma>0$ and $a(u+\sigma s)\geq au$ because $as\geq 0$. By Assumption $ax<\infty$ for all $x\in\mathcal{S}$ thus $as=0$, i.e. the search direction stands perpendicular on the gradient of the objective function. Choose

$$\begin{aligned} \nu &:= \arg\min\{f(j):=\frac{\gamma^j-b^ju}{-b^js},\ b^js<0,\ j=k+1,\dots,m\}\\ \sigma &:= f(\nu)>0. \end{aligned}$$

Without loss of generality, let $\nu=k+1$. Then as above

$$\begin{aligned} b^{k+1}(u-\sigma s) &= b^{k+1}u-\frac{\gamma^{k+1}-b^{k+1}u}{-b^{k+1}s}b^{k+1}s=\gamma^{k+1}\\ b^j(u-\sigma s) &< \gamma^j, \quad j=k+2,\dots,m,\\ b^j(u-\sigma s) &= \gamma^j, \quad j=1,\dots,k,\ \text{weil } b^js=0. \end{aligned}$$

Therefore $v=u-\sigma s\in\mathcal{S}$ and $a(u-\sigma s)=au$ because $as=0$. Accordingly $k+1$ conditions are active in $v=u-\sigma s\in\mathcal{S}$ and $av=au$. Also in this case b^{k+1} is linearly independent of $b^j,\ j=1,\dots,k$, because

$$b^{k+1}s<0 \text{ and } b^js=0,\ j=1,\dots,k.$$

Continuation of this process yields finally a $x\in\mathcal{S}$ with n active side conditions of which the gradients are linearly independent hence a corner point. Since Ω has only a finite number of such points, we obtain the assertion by this way.