Supplements 1 to Chapter II

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To Section 2.1

Theorem 1 *(Theorem 2.2, Cauchy's Error Representation) Let the function f be (n+1)-times differentiable in $[a,b]$ an let $[u,v,\dots,w]$ be the smallest interval $\mathcal{I}\subset\mathbb{R}$ containing all $u,v,\dots,w\in\mathcal{I}$. Then $\forall\, x\in[a,\,b]\ \ \exists\,\xi_x\in[x_0,\dots,x_n,x]$:*

$$f(x)-p_n(x;f)=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\,\omega(x)\,,\quad \omega(x)=(x-x_0)\cdots(x-x_n)\,. \tag{1}$$

Proof. Let $c(x)=[f(x)-p_n(x,f)]/\omega(x),\ x\neq x_i,\ x$ be fixed (!), and let

$$F(t)=f(t)-p_n(t,f)-c(x)\omega(t),\quad \omega\in\Pi_{n+1}\,.$$

F has at least $n+2$ roots $x_0,\dots,x_n,x$. By Rolle's Theorem

$$\begin{array}{lll} F'(t) & \text{at least } n+2-1=n+1 & \text{roots,}\\ \cdots & \cdots & \\ F^{(n+1)}(t) & \text{at least } n+2-(n+1)=1 & \text{roots } \xi_x \text{ in } [x_0,\dots,x_n,x]\,, \end{array}$$

therefore

$$\begin{array}{lll} F^{(n+1)}(\xi_x) & = & f^{(n+1)}(\xi_x)-[(n+1)!]c(x)=0\\ c(x) & = & f^{(n+1)}(\xi_x)/(n+1)!\,. \end{array}$$

□

B-Splines Let Π_n be the set of all polynomials of degree Grad $\le n$. Let $\Omega:=\{x_i\}_{i\in\mathbb{Z}},\ x_i\in\mathbb{R}$, be a weakly monotonically increasing *node sequence*. Ω defines a *partition* Δ_m of the interval $[a,b]\subset\mathbb{R}$, if $x_0=a$ and $x_m=b$,

$$\ldots\le a=x_0\le x_1\le\ldots\le x_m=b\le\ldots.$$

In the present context, Ω is called *non-confluent* if all x_i are mutually distinct and confluent else.

Definition 1 *Two sufficiently smooth functions f and g coincide on the subset $\Phi\subset\Omega$ if*

$$f^{(j-1)}(x)=g^{(j-1)}(x),\quad j=1,\dots,k,$$

for all x which appear in Φ exactly k-times.

Let $p_{0,\dots,m}(x;f)\in\Pi_m$ be the interpolating polynomial which coincides with f on $\{x_0,\dots,x_m\}$,

$$p_{0,\dots,m}(x;f)=f(x_0)+\sum_{i=1}^{m}(x-x_0)\cdots(x-x_{i-1})[x_0,\dots,x_i]f \tag{2}$$

$((x-x_{-1})=1)$. Then the error satisfies

$$f(x)=p_{0,\dots,m}(x;f)+(x-x_0)\cdots(x-x_m)[x_0,\dots,x_m,x]f, \tag{3}$$

because the polynomial on the right side has the value $f(x)$ in every point x for fixed x. The highest term

$$[x_0,\dots,x_m]f \tag{4}$$

is called (generalized) *divided difference* of order m and $[x_i,\dots,x_k]f$ is defined correspondingly such that we can confine ourselves to (4) in the sequel.

Lemma 1 *Es gilt*
(a) $[x_0,\dots,x_m]f$ *is independent of the succession of the nodes* x_i.
(b) $[x_0,\dots,x_m]f$ *is linear in* f.
(c)
$$[x_0,\dots,x_m]f=\frac{f^{(m)}(\xi)}{m!},\ \xi\in[x_0,\dots,x_m],$$
in particular, this relation holds with $\xi=x$ *if* $x_0=\ldots=x_m=x$.
(d) LEIBNIZ*' rule for* $f:x\mapsto g(x)\cdot h(x)$,
$$[x_0,\dots,x_m]f=\sum_{k=0}^{m}([x_0,\dots,x_k]g)([x_k,\dots,x_m]h).$$
(e)
$$[x_0,\dots,x_m]f=\frac{[x_0,\dots,x_{r-1},x_{r+1},\dots,x_m]f-[x_0,\dots,x_{s-1},x_{s+1},\dots,x_m]f}{x_s-x_r}$$
if $x_s\neq x_r$.

Proof. (a) and (b) follow from the existence and uniqueness theorem for polynomial interpolation.
(c) follows from (3) and the error representation after CAUCHY
$$f(x)=p_{0,\dots,m}(x;f)+\frac{f^{(m+1)}(\xi)}{(m+1)!}(x-x_0)\cdots(x-x_m).$$
(d) (Cf. [DeBoor], p. 5.) We consider the function
$$\begin{aligned}F(x)\ &=\sum_{r=0}^{m}(x-x_0)\dots(x-x_{r-1})[x_0,\dots,x_r]g\\ &\times\sum_{s=0}^{m}(x-x_{s+1})\cdots(x-x_m)[x_s,\dots,x_m]h,\end{aligned}$$
($(x-x_{m+1})=1$) which coincides with f at the points $x_0,\dots,x_m$ since, by (2), the first factor coincides with g and the second with h at these points. Expansion yields
$$F(x)=\sum_{r,s=0}^{m}\dots=\sum_{r\le s}\dots+\sum_{r>s}\dots.$$
But the sum $\sum_{r>s}\dots$ disappears at the points $x_0,\dots,x_m$ because each term contains all factors $(x-x_i)$, $i=0,\dots,m$. Accordingly, $\sum_{r\le s}\dots$ must coincide with f at the points $x_0,\dots,x_m$ as well. The $\sum_{r\le s}\dots$ is now a polynomial of degree m with highest term
$$\sum_{r=s}([x_0,\dots,x_r]g)([x_s,\dots,x_m]h),$$
which, by uniqueness, must coincide with the highest term $[x_0,\dots,x_m]f$ of the interpolating polynomial of degree m at the points $x_0,\dots,x_m$.
(e) See Höllig: Numerische Mathematik §3.1.

Definition 2 *A function* $s:[a,b]\to\mathbb{R}$ *is a polynomial spline or briefly* **spline** *of degree* n *w.r.t. the partition* Δ_m *if*
(a) $s\in C^{n-1}[a,b]$,
(b) $s\in\Pi_n$ *in* $[x_i,x_{i+1})$, $i=0,\dots,m-1$.

The set of these splines is denoted by $S_n(\Delta_m)$.
For $k \in \mathbb{N}_0$ let

$$\begin{aligned} p_k(x) &:= x^k, \\ q_k(t,x) &:= (t-x)_+^k := \mathrm{Max}\{(t-x)^k, 0\} \quad \text{(FÖPPL symbol)}. \end{aligned}$$

$q_k(t,x)$ has $k-1$ continuous derivatives w.r.t. both arguments and the k-th derivative makes a jump of height $k!$ resp. $(-1)^k k!$.

Theorem 2 *The set $S_n(\Delta_m)$ is a linear space of dimension $m+n$. The elements*

$$p_0, \ldots, p_n, q_n(\,\cdot\,, x_1), \ldots, q_n(\,\cdot\,, x_{m-1})$$

form a basis of $S_n(\Delta_m)$.

Proof. See [Hämmerlin], p. 246.

Definition 3

$$B_{i,n}(x) := (x_{i+n+1} - x_i)[x_i, \ldots, x_{i+n+1}]q_n(\,\cdot\,, x)$$

is the i-th normalized B-spline *(B like "b"asis) of degree n w.r.t. the node sequence Ω.*

Properties:

partition of unity	$\sum_{i\in\mathbb{Z}} B_{i,n}(x) = 1,\ x \in \mathbb{R}$,
positivity	$B_{i,n}(x) \geq 0$,
lokal support	$B_{i,n}(x) = 0,\ x \notin [x_i, x_{i+n+1}]$,
continuity	$B_{i,n}(x)$ $(n-1)$-times continuously differentiable.

Proof [Hämmerlin], §6.3.

Lemma 2 *(Recurrence Formula for B-splines)*

$$\begin{aligned} B_{i,n}(x) &= w_{i,n}(x)B_{i,n-1}(x) + (1 - w_{i+1,n}(x))B_{i+1,n-1}(x), \\ w_{i,n}(x) &= \begin{cases} \dfrac{x - x_i}{x_{i+n} - x_i}, & x_{i+n} \neq x_i, \\ \textit{arbitrary.} & \textit{else}, \end{cases} \\ B_{i,0}(x) &= \begin{cases} 1, & x_i \leq x < x_{i+1}, \\ 0 & \textit{else.} \end{cases} \end{aligned}$$

Proof. Cf. [DeBoor], p. 130. By definition, we have

$$\begin{aligned} [x_i](\,\cdot - x) &= (x_i - x), \\ [x_i, x_{i+1}](\,\cdot - x) &= 1, \\ [x_i, \ldots, x_{i+k}]q_1(\,\cdot - x) &= 0,\ k > 1, \\ q_n(t,x) &= (t-x)q_{n-1}(t,x). \end{aligned}$$

By application of these relations we obtain the LEIBNIZ rule

$$\begin{aligned} &[x_i, \ldots, x_{i+n+1}]q_n(\,\cdot\,, x) = [x_i, \ldots, x_{i+n+1}]((\,\cdot - x)q_{n-1}(\,\cdot\,, x)) \\ &= \sum_{r=i}^{i+n+1} ([x_i, \ldots, x_{i+r}](\,\cdot - x))([x_{i+r}, \ldots, x_{i+n+1}]q_{n-1}(\,\cdot\,, x) \\ &= (x_i - x)[x_i, \ldots, x_{i+n+1}]q_{n-1}(\,\cdot\,, x) + 1 \cdot [x_{i+1}, \ldots, x_{i+n+1}]q_{n-1}(\,\cdot\,, x) \end{aligned}$$

and by Lemma 1(e)

$$(x_i - x)[x_i, \ldots, x_{i+n+1}]f = \frac{x_i - x}{x_{i+n+1} - x_i}\left([x_{i+1}, \ldots, x_{i+n+1}]f - [x_i, \ldots, x_{i+n}]f\right).$$

Together we obtain

$$\begin{aligned}
&[x_i, \ldots, x_{i+n+1}]q_n(\,\cdot\,, x)\\
&= \frac{x_i - x}{x_{i+n+1} - x_i}[x_{i+1}, \ldots, x_{i+n+1}]q_{n-1}(\,\cdot\,, x) - \frac{x_i - x}{x_{i+n+1} - x_i}[x_i, \ldots, x_{i+n}]q_{n-1}(\,\cdot\,, x)\\
&+ \frac{x_{i+n+1} - x_i}{x_{i+n+1} - x_i}[x_{i+1}, \ldots x_{i+n+1}]q_{n-1}(\,\cdot\,, x)\\
&= \frac{x - x_i}{x_{i+n+1} - x_i}[x_i, \ldots, x_{i+n}]q_{n-1}(\,\cdot\,, x) + \frac{x_{i+n+1} - x}{x_{i+n+1} - x_i}[x_{i+1}, \ldots, x_{i+n+1}]q_{n-1}(\,\cdot\,, x).
\end{aligned}$$

After multiplication by $(x_{i+d+1} - x_i)$ we obtain by this way the desired recurrence formula

$$B_{i,n}(x) = \frac{x - x_i}{x_{i+n} - x_i}B_{i,n-1}(x) + \frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i+1}}B_{i+1,n-1}(x).$$

Theorem 3 *(Representation Theorem) The B-splines*

$$B_{-n,n}, \ldots, B_{m-1,n}$$

form a basis of the space of splines $S_n(\Delta_m)$ *.*

By this result every spline $s \in S_n(\Delta_m)$ has the representation

$$s(x) = \sum_{i=-n}^{m-1} \underline{a}_i B_{i,n}(x). \tag{5}$$

Choosing $\underline{a}_i \in \mathbb{R}^2$ or $\underline{a}_i \in \mathbb{R}^3$ there results the corresponding spline curve in the plane resp. in the space. The polygon with corners $\underline{a}_i$ is called *control polygon* or DeBoor polygon.
The following *algorithm of* DeBoor is used for the calculation of a spline $s \in S_n(\Delta_m)$ at the point $x \in [x_l, x_{l+1}) \subset [x_0, x_m]$.

$$\begin{aligned}
\underline{a}^k{}_i &= \alpha^k{}_i \underline{a}^{k-1}{}_i + (1 - \alpha^k{}_i)\underline{a}^{k-1}{}_{i-1},\ \alpha^k{}_i = \frac{x - x_i}{x_{i+n+1-k} - x_i},\ i \in \{l-d, \ldots, l\},\\
\underline{a}^0{}_i &= \underline{a}_i,\ \underline{a}^n{}_l = s(x).
\end{aligned}$$

Proof. Cf. [DeBoor], p. 146 ff. By the recurrence formula

$$\begin{aligned}
s(x) &= \textstyle\sum_i a_i B_{i,n}(x)\\
&= \sum_i a_i \frac{x - x_i}{x_{i+n} - x_i}B_{i,n-1}(x) + \sum_i a_i \frac{x_{i+n} - x}{x_{i+n} - x_i}B_{i+1,n-1}(x).
\end{aligned}$$

We write $i = j - 1$ in the xecond sum and combine both sums then

$$s(x) = \sum_i a_i^1(x) B_{i,n-1}(x)$$

where

$$a_i^1(x) = \frac{(x - x_i)a_i + (x_{i+n-1} - x)a_{i-1}}{x_{i+n-1} - x_i}.$$

Repeting this step for $k = 2, \ldots, n$ yields

$$s(x) = \sum_i {a^k}_i(x) B_{i,n-k}(x)$$

where

$${a^k}_i(x) = \frac{(x - x_i){a^{k-1}}_i(x) + (x_{i+n-k} - x){a^k}_{i-1}(x)}{x_{i+n-j} - x_i}, \ k > 0.$$

DeBoor scheme

$$\begin{array}{lllll} a^0_{l-n} & & & & \\ a^0_{l-n+1} & a^1_{l-n+1}(x) & & & \\ a^0_{l-n+2} & a^1_{l-n+2}(x) & a^2_{l-n+2}(x) & & \\ \ldots & \ldots & \ldots & & \\ a^0_l & a^1_l & \ldots & \ldots & a^n_l(x) = s(x) \end{array}$$

So one needs the node sequence $\{\underline{a}_{-n}, \ldots, \underline{a}_{m-1}\}$ to calculate the spline $s(x)$ at an arbitrary point $x \in [x_0, x_m]$.

To Section 2.2 Main theorem on orthogonal polynomials.

Theorem 4 *(Existence and Construction)* (1°) *Adopting Assumption 2.1* $\forall\, i \in \mathbb{N}_0 \ \exists!\ p_i \in \overline{\Pi}_i : i \neq k \implies (p_i, p_k) = 0$.
(2°) *The orthogonal polynomials are uniquely determined by the three-term recurrence relation (with $xp : x \mapsto xp(x)$)*

$$\begin{aligned} &p_{-1}(x) = 0\,, \ \ p_0(x) = 1\,, \ \ p_{i+1}(x) = (x - \delta_{i+1})p_i(x) - \gamma_{i+1}^2 p_{i-1}(x)\,, \ i \geq 0, \\ &\delta_{i+1} = (xp_i, p_i)/(p_i, p_i), \ i \geq 0\,, \ \ \gamma_{i+1}^2 = \begin{cases} 0, & i = 0, \\ (p_i, p_i)/(p_{i-1}, p_{i-1}), & i \geq 1. \end{cases} \end{aligned} \tag{6}$$

Proof. By Gram-Schmidt *orthogonalization* of the monoms $x \mapsto x^k$. The assertion is clear for $p_0(x) \equiv 1$. Let the assertion be true for all polynomials of degree $j \leq i$. Then it is to be shown that a $p_{i+1} \in \overline{\Pi}_{i+1}$ exists such that $(p_{i+1}, p_j) = 0\,, \ j \leq i$, and that p_{i+1} has the above properties. By the fundamental theorem of algebra or by direct verification using the norm it is shown that the polynomials $p_j, \ j = 0 : i$, are linear independent. Therefore for $p_{i+1} \in \overline{\Pi}_{i+1}$ uniquely

$$p_{i+1}(x) = (x - \delta_{i+1})p_i(x) + c_{i-1}p_{i-1}(x) + \ldots + c_0 p_0(x).$$

Because $(p_j, p_k) = 0, \ \forall\, j, k \leq i, \ j \neq k$, $(p_{i+1}, p_j) = 0$ for all $j \leq i$ if and only if the following bothe equations are fulfilled:

$$\begin{array}{lclcll} (p_{i+1}, p_i) & = & (xp_i, p_i) - \delta_{i+1}(p_i, p_i) & = & 0, & j = i, \\ (p_{i+1}, p_{j-1}) & = & (xp_{j-1}, p_i) + c_{j-1}(p_{j-1}, p_{j-1}) & = & 0, & \forall\, j \leq i. \end{array} \tag{7}$$

By Assumption 2.1 we have $(p_k, p_k) \neq 0, \ k \leq i$, therefore the assertion follows for δ_{i+1} by (7). By inductin hypothesis

$$\begin{array}{rlcll} & p_j(x) & = & (x - \delta_j)p_{j-1}(x) - \gamma_j^2 p_{j-2}(x), & j \leq i, \\ \Longrightarrow & (p_j, p_i) & = & (xp_{j-1}, p_i) - \delta_j(p_{j-1}, p_i) - \gamma_j^2(p_{j-2}, p_i), & j \leq i, \\ \Longrightarrow & (p_j, p_i) & = & (xp_{j-1}, p_i), & j \leq i, \\ \Longrightarrow & c_{j-1} & = & -\dfrac{(p_j, p_i)}{(p_{j-1}, p_{j-1})} = \begin{cases} -\gamma_{i+1}^2, & j = i \\ 0, & j < i \end{cases}. & \end{array}$$

□

Theorem 5 *For $\delta, \varepsilon \in \{0,\, 1\}$, there exists a unique integration rule*

$$\int_0^1 f(x)\,dx \approx \delta\beta_0\, f(0) + \underline{b}^T F(\underline{x}) + \varepsilon\, \beta_{n+1} f(1) \tag{8}$$

of maximum degree $\widetilde{N} = 2n + \delta + \varepsilon - 1$.

Proof. Let $\underline{e} \in \mathbb{R}^n$ be the vector consisting of units only.
(1°) For all $\underline{b} \in \mathbb{R}^n$ with $\underline{b}^T \underline{e} = 1$ there exist uniquely weights β_0, β_{n+1} such that (8) has order $N^* \geq \delta + \varepsilon - 1$:
For $\delta = \varepsilon = 0$ we have $\underline{b}^T(\gamma\, \underline{e}) = \gamma$ hence $N^* \geq 0 > -1$.
For $(\delta, \varepsilon) = (1, 0)$ or $(\delta, \varepsilon) = (0, 1)$ and $f(x) \equiv 1$, it follows that $1 = \delta\beta_0 f(0) + \underline{b}^T \underline{e} + \varepsilon\beta_{n+1} f(1)$, which yields β_0 resp. β_{n+1}, and $N^* \geq 0$.
For $\delta = \varepsilon = 1$ we have

$$\begin{array}{rclcccccl} f(x) & = & x & \Longrightarrow & \dfrac{1}{2} & = & 0 + \underline{b}^T \underline{x} + \beta_{n+1} & \Longrightarrow & \beta_{n+1} \\ f(x) & = & 1 & \Longrightarrow & 1 & = & \beta_0 + \underline{b}^T \underline{e} + \beta_{n+1} & \Longrightarrow & \beta_0\,, \end{array}$$

and therefore $N^* = 1$.
(2°) Choose Gauss weights and Gauss nodes $\widetilde{\underline{b}}$, $\underline{x}$ by Theorem 2.3.2 w.r.t. the weight function $\omega^*(t) = t^\delta (1 - t)^\varepsilon$ in $[0,\, 1]$ and insert
$\underline{b} = [\widetilde{\beta}_i / \omega^*(x_i)]_{i=1}^n$. For β_0, β_{n+1} by (1°), the integration rule (8) then has order $\widetilde{N} = 2n - 1 + \delta + \varepsilon$: Division by $\omega^*(t)$ yields for $p \in \Pi_{\widetilde{N}}$

$$\begin{array}{rcccccccl} & p(t) & = & p_1(t) & + & \omega^*(t) & \cdot & p_2(t) & := p_1(t) + q(t) \\ \text{order:} & \widetilde{N} & & \delta + \varepsilon - 1 & & \delta + \varepsilon & + & 2n - 1 & \end{array}$$

$p_1(t)$ is integrated exactly by (1°), and $q(t) \in \Pi_{\widetilde{N}}$ is integrated exactly because

$$\begin{aligned} \int_0^1 q(t)\, dt &= \int_0^1 \omega^*(t) p_2(t)\, dt = \sum_{i=1}^n \widetilde{\beta}_i p_2(x_i) \\ &= \sum_{i=1}^n \frac{\widetilde{\beta}_i}{\omega^*(x_i)} \omega^*(x_i) p_2(x_i) = \sum_{i=1}^n \beta_i q(x_i) = 0 + \underline{b}^T Q(\underline{x}) + 0\,. \end{aligned}$$

Accordingly, p is integrated exactly by linearity.
(3°) For $\delta = \varepsilon = 0$, $\widetilde{N}$ is maximum by Theorem 2.3.2, otherwise choose

$$p(x) = x^\delta (1 - x)^\varepsilon \prod_{i=1}^n (x - x_i)^2 \in \Pi_{2n+\delta+\varepsilon 1}$$

In all cases $(\delta, \varepsilon) = (1, 0)\,,\, (0, 1)\,,\, (1, 1)$ the exact integral is positive whereas the integration rule has the value zero hence the error is non-zero. □