Supplements 2 to Chapter I

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To Section 1.9 Let $\mathcal{E}, \mathcal{F}$ be BANACH spaces and $\mathcal{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Theorem 1 *(Theorem 1.19, Bordering Lemma) Let the linear operator* $E : \mathcal{E} \times \mathcal{K}^r \to \mathcal{F} \times \mathcal{K}^r$ *be of the form*

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with

$$\begin{array}{ll} A : \mathcal{E} \to \mathcal{F}, & B : \mathcal{K}^r \to \mathcal{F} \\ C : \mathcal{E} \to \mathcal{K}^r, & D : \mathcal{K}^r \to \mathcal{K}^r. \end{array}$$

(B1) If A is bijective then E is bijective iff $D - CA^{-1}B$ is bijective.
(B2) If A is not bijective and $\dim \mathrm{Ker}(A) = \mathrm{codim}\, \mathrm{Range}(A) = r \geq 1$ *then E is bijective iff*

$$\begin{array}{ll} (B21)\ \dim \mathrm{Range}(B) = r, & (B22)\ \mathrm{Range}(B) \cap \mathrm{Range}(A) = \{0\}, \\ (B23)\ \dim \mathrm{Range}(C) = r, & (B24)\ Ker(A) \cap Ker(C) = \{0\}. \end{array}$$

(B3) If A is not bijective and $\dim \mathrm{Ker}(A) > r$ *then E is not bijective.*

Proof. See [Decker], pp. 428–430. We consider the linear system

$$\begin{array}{l} Ax + Bv = y \\ Cx + Dv = w \end{array}, \tag{1}$$

where $(x, y) \in \mathcal{E} \times \mathcal{F}$ and $(v, w) \in \mathcal{K}^r \times \mathcal{K}^r$. Then E is bijective iff $\forall\ (y, w) \in \mathcal{F} \times \mathcal{K}^r\ \exists\ (x, v) \in \mathcal{E} \times \mathcal{K}^r$ such that (1) holds and if $(x, v) = (0, 0)$ is the only solution of this system with right side $(y, w) = (0, 0)$.
Case (B1). Suppose that A is bijective then A^{-1} exists and (1) is formally equivalent to the linear system

$$\begin{array}{l} x = A^{-1}(y - Bv) \\ (D - CA^{-1}B)v = w - CA^{-1}y. \end{array} \tag{2}$$

Considering these equations, we can see that E is injective iff $D - CA^{-1}B : \mathcal{K}^r \to \mathcal{K}^r$ is injective. From Linear Algebra we also know that $D - CA^{-1}B$ is injective iff it is bijective, and bijectivity of this matrix implies bijectivity of E. This proves the first part of the theorem. In fact, we have in case (B1) the explicit formula

$$E^{-1} = \begin{bmatrix} A^{-1}[I + B(D - CA^{-1}B)^{-1}CA^{-1}] & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

Case (B2). Suppose that

$$\dim \mathrm{Ker}(A) = \mathrm{codim}\, \mathrm{Range}(A) = r \geq 1 \tag{3}$$

– then A is not bijective – and let first (B21) – (B24) be fulfilled. We show again that E is bijective. From (3), $r = \dim \mathrm{Range}(B)$, and $\mathrm{Range}(A) \cap \mathrm{Range}(B) = \{0\}$ we see that

$$\mathcal{F} = \mathrm{Range}(A) \oplus \mathrm{Range}(B), \tag{4}$$

and that B is injective. Also, from Range$(C) = \mathcal{K}^r$, $dim\, Ker(A) = r$, and Ker$(A) \cap$ Ker$(C) = \{0\}$ we see that

$$\mathcal{E} = \text{Ker}(A) \oplus \text{Ker}(C) \tag{5}$$

hence

$$C \mid_{\text{Ker}(A)} \text{ bijective} \quad \text{and } A \mid_{\text{Ker}(C)} \text{ injective.}$$

Now, by (5),

$$\forall\, x \in \mathcal{E}\ \exists!\ (x_A, x_C) \in \text{Ker}(A) \times \text{Ker}(C) : x = x_A + x_C.$$

Inserting this direct decomposition of x into (1) we obtain

$$\begin{array}{l} y = Ax_C + Bv \\ w = Cx_A + Dv \end{array}, \tag{6}$$

hence, by (4), $\forall\ y \in \mathcal{F}\ \exists!\ (x_c(y), v(y)) \in \text{Ker}(C) \times \mathcal{K}^r$ solving (6)(i). Furthermore, (6)(ii) is equivalent to

$$x_A = [C \mid_{\text{Ker}(A)}]^{-1}(w - Dv)$$

such that $x_A \equiv x_A(y, w)$ hence E is surjective.
If we insert $y = 0$ into (6)(i) then we get $x_C = 0$ and $v = 0$ by (4). If then in addition $w = 0$, the second equation in (6) implies that $x_A = 0$ since $v = 0$ and $C \mid_{\text{Ker}(A)}$ is bijective. Therefore E is injective. So E is bijective and (B2.1) – (B2.4) is a sufficient condition.
To show necessitiy, let E be bijective while (3) holds. Now (1) implies that

$$\forall\, y \in \mathcal{F}\ \exists\ (x, v) \in \mathcal{E} \times \mathcal{K}^r : y - Bv = Ax \in \text{Range}(A)$$

hence the direct decomposition

$$\mathcal{F} = \text{Range}(A) \oplus \mathcal{V},$$

where $dim\, \mathcal{V} = \text{codim(Range}(A)) = r$, shows that $\mathcal{V} \subset \text{Range}(B)$. Therefore we have

$$r = \dim \text{Range}(\mathcal{V}) \leq \dim \text{Range}(B) \leq \dim \mathcal{K}^r = r$$

hence B is injective and $\mathcal{V} = \text{Range}(B)$, i.e., (B21) and (B22) are fulfilled.
Let Ker$(A) = span\{x_1, \ldots, x_r\} \subset \mathcal{E}$ and note that

$$E \begin{bmatrix} x_j \\ 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_j \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ w_j \end{bmatrix}, \quad w_j := Cx_j, \quad j = 1, \ldots, r.$$

As the linear operator E is bijective, the elements $w_1, \ldots, w_r$ must be linear independent and hence the linear operator $C : \mathcal{E} \to \mathcal{K}^r$ is surjective, resp. (B23) holds.
If (B24) fails to hold, i.e., Ker$(A) \cap$ Ker$(C) \neq \{0\}$, then we have $Ax_0 = 0 = Cx_0$ for some $0 \neq x_0 \in \mathcal{E}$. Thus $(x, v) = (x_0, 0)$ is a nontrivial solution of $E(x, v) = (0, 0)$ in contradiction to the assumption that E is injective hence finally (B24) holds, too.

Case (B3). If $\dim \operatorname{Ker}(A) > r$ then we can choose $x_1, \dots, x_{r+1}$ linear independent such that $Ax_i = 0$. Since Range$(C) \subset \mathcal{K}^r$, the elements $Cx_1, \dots, Cx_{r+1}$ must be linear independent. So there exists a (r+1)–tuple $\{\alpha_1, \dots, \alpha_{r+1}\}$ with not all components disappearing such that

$$0 \neq x := \sum_{j=1}^{r+1} \alpha_j x_j$$

satisfies $Ax = 0 \in \mathcal{F}$ and $Cx = 0 \in \mathcal{K}^r$ hence $E(x, 0) = (0, 0)$ and E is not bijective.

Remarks. (i) If A is bounded then E is bounded and if E^{-1} exists then it is also bounded.
(ii) If $r = 1$ and $\operatorname{Ker}(A) = [\operatorname{Range}(A^*)]^\perp$ then the regularity condition (ii) simply reduces to

$$B \notin \operatorname{Range}(A), \quad C \notin \operatorname{Range}(A^*), \tag{7}$$

which is equivalent to

$$u^* B \neq 0, \quad C^* v \neq 0,$$

for $A^* u^* = 0$, $Av = 0$, $u^*(v) \neq 0$.
(iii) In this volume we need only the first part (B1) of the Bordering Lemma. Fundamental tools in Nonlinear Analysis are the following theorem of Banach and the Implicit Function Theorem.

Theorem 2 *(Contraction Mapping Theorem) Let $\mathcal{U} \subset \mathcal{E}$ be a closed subset and $f : \mathcal{U} \to \mathcal{U}$ a contraction on $\mathcal{U}$. Then:*
(i) There exists a unique $y^* \in \mathcal{U}$ such that $y^* = f(y^*)$.
(ii) Let $y_0 \in \mathcal{U}$ be arbitrary and

$$y_{n+1} = f(y_n), \ n = 0, 1, 2, \dots,$$

then $y^* = \lim_{n\to\infty} y_n$.
(iii)

$$\|y^* - y_n\| \leq \frac{\alpha^n}{1-\alpha} \|y_1 - y_0\|$$

(a-posteriori error bound).

Proof. See e.g. [Chow], chap. II.

Definition 1 *Let $\mathcal{U} \subset \mathcal{E}$, $\mathcal{V} \subset \mathcal{F}$ be open then $f : \mathcal{U} \to \mathcal{V}$ is a C^r-diffeomorphism $(r \geq 1)$ if f is bijective and f, f^{-1} are both r-times continuously differentiable.*

Theorem 3 *(Theorem 1.21, Inverse Mapping Theorem) Let $\mathcal{U} \subset \mathcal{E}$ and $\mathcal{V} \subset \mathcal{F}$ be open and let $a \in \mathcal{U}$ be fixed. Let $F \in C^r(\mathcal{U}, \mathcal{V})$, $r \geq 1$, and let $F'(a) \in \mathcal{GL}(\mathcal{E}, \mathcal{F})$ (Fréchet-derivative). Then there exist open subsets $a \in \mathcal{U}' \subset \mathcal{U}$ and nonempty $\mathcal{V}' \subset \mathcal{V}$ such that the restriction of F onto $\mathcal{U}'$ is a C^r-diffeomorphism.*

Proof. (i) It suffices to consider the case $\mathcal{E} = \mathcal{F}$ and the mapping

$$f : \mathcal{E} \ni x \mapsto F'(a)^{-1}[F(a + x) - F(a)]$$

with $f(0)=0$ and $f'(0)=id$ (identity).
(ii) The function

$$g : x \mapsto x - f(x)$$

then has the properties $g(0)=0$ and $g'(0)=0$. Therefore, there exists a $\delta > 0$ such that $K_\delta := \{x \in \mathcal{E}, \|x\| < \delta\} \subset \mathcal{U}$ and $\forall\, x \in K_\delta\ \|g'(x)\| \leq 1/2$, hence

$$\forall\, x \in K_\delta \quad \| \int_0^1 g'(tx)x dt\| \leq \frac{1}{2}\|x\|.$$

Let now $y \in \overline{K}_{\delta/2}$ be fixed and consider the mapping

$$G(\cdot, y) : x \mapsto y + g(x) = y + x - f(x).$$

Then x is a fixed point of G iff $y = f(x)$ and G maps $\overline{K}_\delta$ into $\overline{K}_\delta$. For $\|y\| \leq \delta/2$, G is a contraction on $\overline{K}_\delta$:

$$\begin{aligned} &\|G(x,y) - G(\tilde{x},y)\| = \|g(x) - g(\tilde{x})\| \\ &= \| \textstyle\int_0^1 g'(\tilde{x} + t(x-\tilde{x}))(x-\tilde{x})dt \leq \|x - \tilde{x}\|/2. \end{aligned}$$

Therefore the Contraction Mapping Theorem can be applied to G:

$$\forall\, y \in \overline{K}_{\delta/2}\ \exists!\ x^* \in \overline{K}_\delta \quad x^* = G(x^*).$$

This proves that f is invertible on $\overline{K}_{\delta/2}$ with

$$f^{-1} : \overline{K}_{\delta/2} \ni y \mapsto x^*(y) \in \overline{K}_\delta.$$

(iii) We choose the open sets $\mathcal{V}' = K_{\delta/2}$ and $\mathcal{U}' = f^{-1}(\mathcal{V}')$ then the restriction of f to $\mathcal{U}'$ is bijective and we have to prove the regularity of the restriction φ of f^{-1} to $\mathcal{V}'$. At first we show that φ is Lipschitz-continuous. But

$$\begin{aligned} x_1 = y_1 + g(x_1) &\Longleftrightarrow y_1 = f(x_1), \\ x_2 = y_2 + g(x_2) &\Longleftrightarrow y_2 = f(x_2), \end{aligned}$$

hence

$$\begin{aligned} &\|x_1 - x_2\| \leq \|y_1 - y_2\| + \|g(x_1) - g(x_2)\| \\ &\leq \|y_1 - y_2\| + \tfrac{1}{2}\|x_1 - x_2\| \end{aligned}$$

for $(x_i, y_i) \in \mathcal{U}' \times \mathcal{V}'$, or

$$\frac{1}{2}\|x_1 - x_2\| \leq \|y_1 - y_2\|$$

or

$$\|\varphi(y_1) - \varphi(y_2)\| \leq 2\|y_1 - y_2\|.$$

(iv) We have $f'(0) = id \in \mathcal{GL}(\mathcal{E},\mathcal{E})$ hence there exists a $\delta_1 > 0$ such that

$$\forall\, x \in K_{\delta_1} \quad f'(x) \in \mathcal{GL}(\mathcal{E},\mathcal{E}) \text{ and } \|f'(x)^{-1}\| \leq M.$$

We then obtain

$$\|\varphi(y_1) - \varphi(y_2) - f'(x_2)^{-1}(y_1 - y_2)\|$$
$$= \|f'(x_2)^{-1}[f'(x_2)(x_1 - x_2) + f(x_2) - f(x_1)]\| \le M\|r(x_1, x_2)\|$$

Because f is differentiable, we have

$$r(x_1, x_2) := f(x_1) - f(x_2) - f'(x_2)(x_1 - x_2) = o(\|x_1 - x_2\|)$$

thus we obtain

$$\frac{\|\varphi(y_1] - \varphi(y_2) - f'(x_2)^{-1}(y_1 - y_2)\|}{\|y_1 - y_2\|}$$

$$\le \frac{M\|r(x_1, x_2)\|}{\|y_1 - y_2\|} \le \frac{2M\|r(x_1, x_2)\|}{\|x_1 - x_2\|} \longrightarrow 0$$

for $\|x_1 - x_2\| \to 0$.
This proves that φ is differentiable with $\varphi'(y) = f'(x)^{-1}$.
(v) The proof of continuity of φ' and possible higher smoothness is left to the reader.

φ inherits the smoothness of f also if $f \in C^\infty$ or if f is analytic.
The Implicit Function Theorem is a simple inference of the Inverse Mapping Theorem:

Corollary 1 *(Corollary 1.6, Implicit Function Theorem) Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be Banach spaces and let $f \in C^r(\mathcal{E} \times \mathcal{F}; \mathcal{G})$, $r \ge 1, c = f(a, b)$, $D_2 f(a, b) \in \mathcal{GL}(\mathcal{F}, \mathcal{G})$. Then there exist open $\mathcal{U}, \mathcal{W}$ with $a \in \mathcal{U} \subset \mathcal{E}$, $c \in \mathcal{W} \subset \mathcal{G}$ and a uniquely determined function $\varphi \in C^r(\mathcal{U} \times \mathcal{W}, \mathcal{F})$ such that*

$$b = \varphi(a, c), \quad \forall\, x \in \mathcal{U},\ \forall\, z \in \mathcal{W}\ z = f(x, \varphi(x, z))$$

and φ is as smooth as f.

Proof: Let

$$F(x, y) := \mathcal{E} \times \mathcal{F} \ni (x, y) \mapsto (x, f(x, y)) \in \mathcal{E} \times \mathcal{G}$$

and $F(a, b) = (a, c)$. We have only to show that F is local invertible. But

$$F'(a, b) = \begin{bmatrix} Id & 0 \\ D_1 f(a, b) & D_2 f(a, b) \end{bmatrix}$$

is invertible and continuous hence

$$F'(a, b) \in \mathcal{GL}(\mathcal{E} \times \mathcal{F}, \mathcal{E} \times \mathcal{G})$$

by the Inverse Operator Theorem.
For $z = 0$ we obtain the following second form of the Implicit Function Theorem:

Corollary 2 *(Corollary 1.7, Implicit Function Theorem) Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be Banach spaces and let $f \in C^r(\mathcal{E} \times \mathcal{F}; \mathcal{G})$, $r \ge 1, f(a, b) = 0$, $D_2 f(a, b) \in \mathcal{GL}(\mathcal{F}, \mathcal{G})$. Then there exist open $\mathcal{U}$ with $a \in \mathcal{U} \subset \mathcal{E}$, and a uniquely determined function $\varphi \in C^r(\mathcal{U}, \mathcal{F})$ such that*

$$\varphi(a) = b, \quad \forall\, x \in \mathcal{U}\ f(x, \varphi(x)) = 0$$

and φ is as smooth as f.

For an other proof see also [Wloka], Th. 20.4.

To Section 1.10

Lemma 1 *(Lemma 1.26) Let $\mathcal{X}$, $\mathcal{Y}$ be normed vector spaces, $\mathcal{C} \subset \mathcal{X}$ convex, $\mathcal{K} \subset \mathcal{Y}$ positive cone, and $f : \mathcal{C} \to \mathcal{Y}$ F-differentiable in $\mathcal{D} \supset \mathcal{C}$ open.*
(1°) *f is $\mathcal{K}$-convex if and only if*

$$\forall\, x, y \in \mathcal{C} : f(y) - f(x) - \nabla f(x)(y - x) \geq 0 \;\; i.e. \;\in \mathcal{K}\,. \tag{8}$$

(2°) *Let $\mathcal{X} = \mathbb{R}^n$ and f two-times F-differentiable and $\mathcal{K}$-convex then*

$$\forall\, y \in \mathcal{X} : \nabla\nabla f(x)[yy] \geq 0 \;\; i.e. \;\in \mathcal{K}\,. \tag{9}$$

(3°) *Let $\mathcal{X} = \mathbb{R}^n$, $\mathcal{Y} = \mathbb{R}$, and f two-times F-differentiable and let (9) hold then f is $\mathcal{K}$-convex.*

Proof. (1°) Let f be $\mathcal{K}$-convex then, by the definition, for $0 < \lambda < 1$ directly

$$f(y) - f(x) - \frac{1}{\lambda}\big[f(x + \lambda(y - x)) - f(x)\big] \in \mathcal{K}\,. \tag{10}$$

This yields (8) for $\lambda \to 0$ if the cone $\mathcal{K}$ is closed. Otherwise let $0 < \eta < \lambda < 1$ and $z = y - x$ then

$$\begin{aligned}
&\eta\Big[-\eta^{-1}\big[f(x + \eta z) - f(x)\big] + \lambda^{-1}\big[f(x + \lambda z) - f(x)\big]\Big] \\
&= -f\left(\frac{\eta}{\lambda}(x + \lambda z) + (1 - \frac{\eta}{\lambda})x\right) + \frac{\eta}{\lambda} f(x + \lambda z) + (1 - \frac{\eta}{\lambda}) f(x) \in \mathcal{K}
\end{aligned}$$

So we obtain for $\eta \to 0$

$$\begin{aligned}
&-\nabla f(x) z + \lambda^{-1}\big[f(x + \lambda z) - f(x)\big] \in \mathcal{K} \\
&\implies \lambda^{-1}\big[f(x + \lambda z) - f(x)\big] \in \mathcal{K} + \nabla f(x) z\,.
\end{aligned} \tag{11}$$

Then, by (10) and (11)

$$f(y) - f(x) \in \mathcal{K} + \lambda^{-1}[f(x + \lambda z) - f(x)] \in \mathcal{K} + \mathcal{K} + \nabla f(x) z$$

and thus (8) because $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$.
Conversely, let (8) hold then choose $u, v \in \mathcal{X}$, $0 < \lambda < 1$, set $x = \lambda u + (1 - \lambda) v$ and

$$\begin{aligned}
y = u :&\quad f(u) \geq f(x) + \nabla f(x)(u - x) \quad | \cdot \lambda, \\
y = v :&\quad f(v) \geq f(x) + \nabla f(x)(v - x) \quad | \cdot (1 - \lambda)\,.
\end{aligned}$$

By addition we obtain

$$\begin{aligned}
\lambda f(u) + (1 - \lambda) f(v) &\geq (\lambda + 1 - \lambda) f(x) + \nabla f(x)(\lambda(u - x) + (1 - \lambda)(v - x)) \\
&= f(x)
\end{aligned}$$

and thus convexity of f by definition of x.
(2°) Let f be two-times F-differentiable then, by addition of

$$f(y) - f(x) \in \mathcal{K} + \nabla f(x)(y - x) \;\text{ and }\; f(x) - f(y) \in \mathcal{K} + \nabla f(y)(x - y)\,,$$

we obtain the inequality

$$0 \in \mathcal{K} + \mathcal{K} - [\nabla f(y) - \nabla f(x)](y-x) \subset \mathcal{K} - \nabla^2 f(x)[y-x, y-x] + o(\|y-x\|^2)\,,$$

which implies (9).
(3°) Conversely, let (9) hold and let $0 < \lambda < 1$ as well as

$$z(\lambda) = x + \lambda(y-x)\,,\ \ \varphi(\lambda) = f(z(\lambda)) - \nabla f(y)(z(\lambda) - x)\,.$$

Then there exist a $\delta \in (0,1)$ by the mean value theorem such that, by (9),

$$f(y) - f(x) - \nabla f(x)(y-x) = \varphi(1) - \varphi(0) = \varphi'(\delta)(1-0)$$

$$= [\nabla f(z(\delta)) - \nabla f(x)](y-x) = \int_0^\delta \nabla^2 f(z(\sigma))[y-x, y-x]\, d\sigma \in \mathcal{K}$$

This equation verifies (8) therefore f is $\mathcal{K}$-konvex by (1°).

To Section 1.11

Lemma 2 *(Lemma 1.32)* (1°) *(Contraction, Continuity) Under the assumption of Projecti0n Theorem 1.26*

$$\forall\, v,\, w \in \mathcal{H} : \|Pv - Pw\| \leq \|v - w\|\,.$$

(2°) *(Linearity) The projection operator P is a linear mapping $P : \mathcal{H} \to \mathcal{U}$ if and only if $\mathcal{U}$ is a linear subspace.*

Proof (1°)

$$\begin{aligned} \|v-w\|^2 &= \|v - P\,v + P\,v - P\,w + P\,w - w\|^2 \\ &= \|v - P\,v + P\,w - w\|^2 + \|P\,v - P\,w\|^2 \\ &\quad -2\,(v - P\,v\,,\, P\,w - P\,v) - 2\,(w - P\,w\,,\, P\,v - P\,w)\,. \end{aligned}$$

Because $y = Pw \in U,\ z = Pv \in U$ and the charakterization theorem it follows that

$$\|v-w\|^2 \geq \|v - P\,v + P\,w - w\|^2 + \|P\,v - P\,w\|^2 \geq \|P\,v - P\,w\|^2\,.$$

(2.1°) Let U be a subspace then by the characterization theorem

$$\forall\, v \in U\ \forall\, w \in H : (w - P\,w\,,\, v) = 0\,.$$

For instance

$$\begin{aligned} &\forall\, v \in U : ((y+z) - P\,(y+z)\,,\, v) = 0\,, \\ &\forall\, v \in U : ((y+z) - (P\,y + P\,z)\,,\, v) = 0\,, \end{aligned}$$

implies that

$$\forall\, v \in U : (P\,(y+z) - (P\,y + P\,z)\,,\, v) = 0\,.$$

Inserting here $v = P(y+z) - (Py + Pz) \in U$ we obtain the assertion for the addition.
(2.2°) Conversely, let P be linear

$$P(\alpha\, u + \beta\, v) = \alpha Pu + \beta Pv,$$

then, because $y = Pu \in U$ and $z = Pv \in U$, we find that $\alpha\, y + \beta\, z \in U$ and thus U is linear.