Supplements 1 to Chapter I

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To Section 1.1

Lemma 1 *(Lemma 1.4) Letting* $\|\underline{x}\|^2 = \underline{x}^T\underline{x}$, *the vector* $\underline{x}^* = A^+\underline{b}$ *satisfies*
(1°) $\forall\ \underline{x} \in \mathbb{R}^n : \|A\underline{x}^* - \underline{b}\| \le \|A\underline{x} - \underline{b}\|$,
(2°) $\|A\underline{x}^* - \underline{b}\| = \|A\underline{x} - \underline{b}\| \implies \|\underline{x}^*\| \le \|\underline{x}\|$.

Proff. Every unitary matrix Q satisfies $\|Qx\| = \|x\|$ where $\|\cdot\| := \|\cdot\|_2$. Therefore

$$\|Ax - b\| = \|USV^H x - b\| = \|SV^H x - U^H b\| = \|Sy - c\|$$

where $V^H x = y$ and $U^H b = c$. Let r be the rank of A, then $\arg\min_y \|Sy - c\|$ is given by

$$y_j = \begin{cases} c_j/\sigma_j & j = 1, \dots, r, \\ \text{arbitrary} & j > r. \end{cases}$$

Let $u_j = Ue_j$ and $v_j = Ve_j$ be the j-th column resp. row U resp. V. Because $x = Vy$ and $c = U^H b$, i.e. $c_j = e_j^T U^H b$ we then obtain

$$\begin{aligned} x &= \textstyle\sum_{j=1}^r Ve_j e_j^T U^H b/\sigma_j + \sum_{j=r+1}^n v_j y_j = VS^+U^H b + \sum_{j=r+1}^n v_j y_j \\ &= A^+ b + \textstyle\sum_{j=r+1}^n v_j y_j . \end{aligned}$$

Since the vectors v_j are orthogonal, $x = A^+ b$ is the solution with minimum norm.

To Section 1.2
Proof of formulas (1.17) and (1.18)

$$\begin{aligned} \int_V (\operatorname{grad} \varphi)^T\, dV &= \oint_{\partial V} \varphi\, d\underline{O} = \oint_{\partial V} \varphi\, \underline{n} dO \in \mathbb{R}^3 , \\ \int_V \operatorname{rot} \underline{v}\, dV &= \oint_{\partial V} d\underline{O} \times \underline{v} = - \oint_{\partial V} \underline{v} \times \underline{n}\, dO \in \mathbb{R}^3 . \end{aligned}$$

and

$$\int_F d\underline{O} \times (\operatorname{grad} \varphi)^T = \oint_{\partial F} \varphi\, d\underline{x} \in \mathbb{R}^3 .$$

Proof. Let $\underline{a}^T\underline{x} = 0$ for all $\underline{a} \in \mathbb{R}^3$ then $\underline{x} = \underline{0}$.
(1°) Let $\underline{a} \in \mathbb{R}^3$ arbitrary constant then $\operatorname{div} \underline{a} = 0$ and $\operatorname{div}(\varphi\underline{a}) = \operatorname{grad}(\varphi) \cdot \underline{a}$. The theorem of Gauss yields for $\underline{v} = \varphi\underline{a}$

$$\int_{\partial V} (\varphi\, \underline{a}) \cdot \underline{n}\, dO = \int_V \operatorname{div}(\varphi\, \underline{a})\, dV = \int_V \operatorname{grad} \varphi \cdot \underline{a}\, dV = [\int_V \operatorname{grad} \varphi\, dV] \cdot \underline{a} ,$$

hence

$$\underline{a} \cdot \int_{\partial V} \varphi\, \underline{n}\, dO = \underline{a} \cdot \int_V (\operatorname{grad} \varphi)^T\, dV .$$

Noe the assertion follows because $\underline{a}$ is arbitrary.
(2°) Recall that

$$\operatorname{div}(\underline{v} \times \underline{w}) = \underline{w} \cdot \operatorname{rot} \underline{v} - \underline{v} \cdot \operatorname{rot} \underline{w} .$$

and $\operatorname{rot}\underline{a} = 0$ for constant $\underline{w} = \underline{a}$ therefore $\operatorname{div}(\underline{v} \times \underline{a}) = \underline{a} \cdot \operatorname{rot}\underline{v}$. With the same argumentation as in (1°) we now obtain the result by the theorem of GAUSS for $\underline{w} = \underline{v} \times \underline{a}$:

$$\int_V \operatorname{div}\underline{w}\,dV = \int_V \operatorname{div}(\underline{v} \times \underline{a})\,dV = \int_V \underline{a} \cdot \operatorname{rot}\underline{v}\,dV = \int_{\partial V} \underline{w} \cdot \underline{n}\,dO$$
$$= \int_{\partial V} (\underline{v} \times \underline{a}) \cdot \underline{n}\,dO = -\underline{a} \cdot \int_{\partial V} (\underline{v} \times \underline{n}\,dO\,.$$

(3°) Recall

$$\operatorname{rot}(\varphi\,\underline{v}) = (\operatorname{grad}\varphi)^T \times \underline{v} + \varphi \operatorname{rot}\underline{v}$$

and $\operatorname{rot}\underline{v} = \underline{0}$ for constant $\underline{v} = \underline{a}$ hence $\operatorname{rot}(\varphi\,\underline{a}) = (\operatorname{grad}\varphi)^T \times \underline{a}$. The theorem of STOKES then yields for $\underline{w} = \varphi\,\underline{a}$

$$\int_F \operatorname{rot}(\varphi\,\underline{a})\cdot\underline{n}\,dO = \int_F ((\operatorname{grad}\varphi)^T \times \underline{a})\cdot\underline{n}\,dO = \underline{a}\cdot\int_F (\underline{n}\times(\operatorname{grad}\varphi)^T)\,dO = \int_{\partial F} \varphi\,\underline{a}\cdot\underline{dx} = \underline{a}\cdot\int \varphi\,\underline{dx}\,.$$

Theorem 1 *(Theorem 1.4, Potential Criterium) Let* $\Omega \subset \mathbb{R}^n$ *be a simply connected domain and* $\underline{v} : \Omega \to \mathbb{R}^n$ *a continuously differentiable vector field. Then there exists a* potential $\varphi : \Omega \to \mathbb{R}$ *with* $\underline{v} = \operatorname{grad}\varphi$ *if and only if* $\operatorname{grad}\underline{v}(\underline{x})$ *is symmetric,*

$$\forall\, \underline{x} \in \Omega : \operatorname{grad}\underline{v}(\underline{x}) = [\operatorname{grad}\underline{v}(\underline{x})]^T\,;$$

this condition is equivalent to $\operatorname{rot}\underline{v}(\underline{x}) = 0$ *for* $n = 2, 3$.

Proof. The right side is necessary by the theorem of H.A.SCHWARZ. We choose $\underline{a} \in G$ fixed, $\underline{x} \in G$ near $\underline{a}$ and for C the straight line from $\underline{a}$ to $\underline{x}$:

$$\underline{w}(t;x) = \underline{a} + t(\underline{x} - \underline{a})\,, \quad 0 \le t \le 1\,.$$

Then by the chain rule (E identity matrix)

$$\operatorname{grad}_x \underline{v}(\underline{w}(t;\underline{x})) = [\operatorname{grad}\underline{v}](\underline{w}(t;\underline{x}))\operatorname{grad}_x \underline{w}(t;\underline{x})$$
$$= [\operatorname{grad}\underline{v}](\underline{w}(t;\underline{x}))tE = t[\operatorname{grad}\underline{v}](\underline{w}(t;\underline{x}))$$

As possible antiderivative we define

$$f(\underline{x}) := \int_0^1 \underline{v}(\underline{w}(t,\underline{x}))^T \dot{\underline{w}}(t;x)\,dt = \int_0^1 \underline{v}(\underline{a} + t(\underline{x} - \underline{a}))^T(\underline{x} - \underline{a})\,dt$$

Then, by $\operatorname{grad}(\underline{v}^T\underline{u}) = \underline{v}^T\operatorname{grad}\underline{u} + \underline{u}^T\operatorname{grad}\underline{v} \in \mathbb{R}_n$ we obtain for $\underline{u}(\underline{x}) = \underline{x}$ that

$$\operatorname{grad} f(\underline{x}) = \int_0^1 \operatorname{grad}_x[\underline{v}(\underline{w}(t;\underline{x}))^T(\underline{x} - \underline{a})]\,dt = \int_0^1 [\underline{v}^T E + (\underline{x} - \underline{a})^T \nabla_x \underline{v}(\underline{w}(t;\underline{x}))t]\,dt$$

If the right side is true then $(\underline{x} - \underline{a})^T\nabla_x\underline{v} = (\underline{x} - \underline{a})^T[\nabla_x\underline{v}]^T$ and thus

$$\operatorname{grad} f(\underline{x}) = \int_0^1 \frac{d}{dt}[t\underline{v}(\underline{w}(t;\underline{x}))^T]\,dt = t\underline{v}(\underline{w}(t;\underline{x}))^T\Big|_0^1$$
$$= 1\cdot\underline{v}(\underline{w}(1;\underline{x}))^T - 0\cdot\underline{v}(\underline{w}(0;\underline{x}))^T = \underline{v}(\underline{w}(1;\underline{x}))^T = \underline{v}(\underline{x})^T \in \mathbb{R}_n\,.$$

Theorem 2 *(Theorem 1.5) Let $\Omega \subset \mathbb{R}^3$ be a star-shaped domain and $\underline{v} : \Omega \to \mathbb{R}^3$ continuously differentiable. Then the vector field $\underline{v}$ has a vector potential $\underline{w}$ with $\underline{v} = \operatorname{rot} \underline{w}$ if and only if $\operatorname{div} \underline{v} = 0$ in Ω.*

Proof. If $\underline{v} = \operatorname{rot} \underline{w}$ then $\operatorname{div} \operatorname{rot} \underline{w} = \operatorname{div} \underline{v} = 0$. Conversely, let without loss of generality

$$\underline{w}(x) = \int_0^1 t(\underline{v}(z + t(x - z)) \times (x - z))\, dt$$

and also without loss of generality $z = 0$. Because

$$\operatorname{rot}(\underline{v} \times \underline{u}) = (\operatorname{div} \underline{u})\underline{v} - (\operatorname{div} \underline{v})\underline{u} + (\operatorname{grad} \underline{v})\underline{u} - (\operatorname{grad} \underline{u})\underline{v}$$

and $\operatorname{div} \underline{v} = 0$, we obtain for $\underline{u}(x) = x$ and $\operatorname{div} x = 3$

$$\begin{aligned}
&\operatorname{rot}_x \int_0^1 t(\underline{v}(tx) \times x)\, dt = \int_0^1 t \operatorname{rot}_x \big(\underline{v}(tx) \times x\big)\, dt \\
&= \int_0^1 t\big[\underline{v}(tx) \operatorname{div} x + t \operatorname{grad} \underline{v}(tx) - E\, \underline{v}(tx)\big]\, dt \quad (E \text{ identity matrix}) \\
&= \int_0^1 \big[2t\underline{v}(tx) + t^2 \frac{d}{dt}\underline{v}(tx)\big)\, dt = \int_0^1 \frac{d}{dt}\big(t^2 \underline{v}(tx)\big]\, dt = \big[t^2 \underline{v}(tx)\big]_0^1 = \underline{v}(x)\,.
\end{aligned}$$

□

Theorem 3 *(Theorem 1.6) Let Ω be "regular" and let $\underline{v} : \Omega \to \mathbb{R}^3$ be a divergece-free vector field then $\underline{v}$ has a divergence-free vector potential.*

Proof. Let there be given a vector potential $\underline{w}_0$ of $\underline{v}$ by Theorem 1.5 then we write $\underline{w} = \underline{w}_0 + \operatorname{grad} \varphi$ and require that $0 = \operatorname{div} \underline{w} = \operatorname{div} \underline{w}_0 + \Delta\varphi$, i.e.

$$\Delta\varphi = -\operatorname{div} \underline{w}_0$$

in Ω. This differantial equation has at least one solution φ. Then $\underline{w}$ is divergence-free and $\underline{v} = \operatorname{rot} \underline{w}$. □

Theorem 4 *(HELMHOLTZ' Decomposition Theorem) Let $\Omega \subset \mathbb{R}^3$ be "regular" and let $\underline{v} : \Omega \to \mathbb{R}^3$ be continuously differentiable then there exists a scalar field φ and a vector field $\underline{w}$ such that*

$$\boxed{\underline{v} = \operatorname{grad} \varphi + \operatorname{rot} \underline{w}}\,.$$

Proof. If there exists a partition of the announced form then applying divergence yields

$$\operatorname{div} \underline{v} = \Delta\varphi,$$

and thus a differential equation for φ. Suppose that φ is a solution of this equation then

$$\operatorname{div}(\underline{v} - \operatorname{grad} \varphi) = 0.$$

By Theorem 1.6 there exists a vector potential $\underline{w}$ such that

$$\underline{v} - \operatorname{grad} \varphi = \operatorname{rot} \underline{w}.$$

To Section 1.3
Deriving of formula (1.24) for the torsion

$$\tau(t)=\frac{\det(\dot{\underline{x}},\ddot{\underline{x}},\dddot{\underline{x}})(t)}{|\dot{\underline{x}}(t)\times\ddot{\underline{x}}(t)|^2}\,.$$

Proof. Since $\dot{\underline{t}}$ parallel to $\underline{n}$ we have $\dot{\underline{t}}\times\underline{n}=0$. Then it follows that

$$|\underline{b}|^2=\underline{b}^T\underline{b}=1 \quad\Longrightarrow\quad \dot{\underline{b}}^T\underline{b}+\underline{b}^T\dot{\underline{b}}=0 \;\Longrightarrow\; \dot{\underline{b}}\;\perp\;\underline{b}\,,$$
$$\dot{\underline{b}}=\dot{\underline{t}}\times\underline{n}+\underline{t}\times\dot{\underline{n}}=\underline{t}\times\dot{\underline{n}} \;\Longrightarrow\; \dot{\underline{b}}\;\perp\;\underline{t}\,,$$

therefore $\dot{\underline{b}}$ must be parallel to $\underline{n}$. Write

$$\frac{\dot{\underline{b}}}{\dot{s}}=-\tau\,\underline{n}\;\left(\text{manchmal auch)}\;\frac{\dot{\underline{b}}}{\dot{s}}=\tau\,\underline{n}\right)\,. \tag{1}$$

From $\underline{b}^T\underline{n}=0$, $\dot{\underline{b}}^T\underline{n}+\underline{b}^T\dot{\underline{n}}=0$, and (1) we obtain

$$\tau\equiv\tau\,\underline{n}\cdot\underline{n}=-\frac{\dot{\underline{b}}\cdot\underline{n}}{\dot{s}}=\frac{\dot{\underline{n}}\cdot\underline{b}}{\dot{s}}=\frac{\dot{\underline{n}}\cdot(\dot{\underline{x}}\times\ddot{\underline{x}})}{\dot{s}|\dot{\underline{x}}\times\ddot{\underline{x}}|}. \tag{2}$$

But $\ddot{\underline{x}}=\ddot{s}\,\underline{t}+\dot{s}^2\kappa\,\underline{n}$ hence

$$\dddot{\underline{x}}=\dddot{s}\,\underline{t}+\ddot{s}\dot{\underline{t}}+2\dot{s}\ddot{s}\kappa\,\underline{n}+\dot{s}^2\dot{\kappa}\,\underline{n}+\dot{s}^2\kappa\,\dot{\underline{n}}\,.$$

Resolution of this equation w.r.t. $\dot{\underline{n}}$ and inserting in (2) yields

$$\tau=\frac{(\dot{\underline{x}}\times\ddot{\underline{x}})^T[\dddot{\underline{x}}-\dddot{s}\,\underline{t}-\ddot{s}\,\dot{\underline{t}}-2\dot{s}\ddot{s}\kappa\,\underline{n}-\dot{s}^2\kappa\,\underline{n}]}{\dot{s}^2\kappa\dot{s}\,|\dot{\underline{x}}\times\ddot{\underline{x}}|}. \tag{3}$$

Note that $\underline{b}$ is parallel to $\dot{\underline{x}}\times\ddot{\underline{x}}$, $\underline{b}\;\perp\;\underline{t}$, $\underline{b}\;\perp\;\underline{n}$, hence also $\underline{b}\;\perp\;\dot{\underline{t}}$. Therefore

$$(\dot{\underline{x}}\times\ddot{\underline{x}})^T[\dddot{s}\,\underline{t}+\ddot{s}\,\dot{\underline{t}}+2\dot{s}\ddot{s}\kappa\,\underline{n}+\dot{s}^2\kappa\,\underline{n}]=0\,.$$

Now, inserting $\kappa=|\dot{\underline{x}}\times\ddot{\underline{x}}|/\dot{s}^3$ in the denominator, the result for τ is obtained.

To Section 1.6

Theorem 5 *(Straightening Theorem) Let* $\underline{v}:\Omega\to\mathbb{R}^n$ *be a conservative vector field, let* $\underline{x}_0\in\Omega$, $\underline{v}(\underline{x}_0)\neq\underline{0}$, *and let* $\underline{e}_1=[1,0,\ldots,0]^T\in\mathbb{R}^n$. *Then there exists an open set* $\mathcal{U}$ *with* $\underline{x}_0\in\mathcal{U}\subset\mathbb{R}^n$ *and a diffeomorphism* $F:\mathcal{U}\to\mathcal{U}$ *such that*

$$\forall\;\underline{x}\in\mathcal{U}:\underline{e}_1=\nabla F(\underline{x})\underline{v}(\underline{x})\,. \tag{4}$$

Proof. For $n=3$. Let $\underline{x}_0=[x_0^1,x_0^2,x_0^3]^T$ and $\underline{v}(\underline{x})=(v^1(\underline{x}),v^2(\underline{x}),v^3(\underline{x}))^T$. Without loss of generality, let $v^1(x_0)\neq 0$. Further, let $x_0^1=0$, else consider the rotated vector field $A^T\underline{v}(A\underline{x})$ with a suitable rotation matrix A. Besides, let

$\underline{X} = [X^1, X^2, X^3]^T \in \mathbb{R}^3$ arbitrary, $\underline{X^*} = [0, X^2, X^3]^T$, and let $\Phi(t, \underline{X})$ be the flux of the vector field $\underline{v}$. We insert $t = X^1 - x_0^1$ and $\underline{X} = \underline{X^*}$ and consider the mapping

$$G : \underline{X} \mapsto \Phi(X^1 - x_0^1, \underline{X^*}) = G(\underline{X}) =: \underline{x} \in \mathbb{R}^3 ,$$

then

$$G(\underline{x_0}) = \Phi(0, [0, x_0^2, x_0^3]^T) = [0, x_0^2, x_0^3]^T = \underline{x_0}$$

because $x_0^1 = 0$ and

$$\frac{\partial}{\partial X^1} G(\underline{X}) = \underline{v}(G(\underline{X})) \quad \text{f"ur } X^1 = x_0^1 .$$

Then, for $X^1 = x_0^1$ because $\Phi(0, \underline{X^*}) = [0, X^2, X^3]^T$,

$$\nabla G(\underline{X}) = \begin{bmatrix} v^1(G(\underline{X})) & 0 & 0 \\ v^2(G(\underline{X})) & 1 & 0 \\ v^3(G(\underline{X})) & 0 & 1 \end{bmatrix} \tag{5}$$

where $G(\underline{x_0}) = \underline{x_0}$, and the matrix $\nabla G(\underline{x_0})$ is regular since $v^1(\underline{x_0}) \neq 0$ by assumption. Since $\underline{v}$ is continuously differentiable, there exists an open neighborhood $\mathcal{U}_0$ where $\underline{x_0} \in \mathcal{U}_0 \subset \Omega$ such that $\nabla G(\underline{X})$ is regular for all $\underline{X} \in \mathcal{U}_0$. By the inverse mapping theorem there exists an open neighborhood $\mathcal{U}$ where $\underline{x_0} \in \mathcal{U} \subset \mathcal{U}_0$ such that G is a diffeomorphism in $\mathcal{U}$. By (5) we have

$$\nabla G(\underline{X})\underline{e_1} = \underline{v}(G(\underline{X})) .$$

Choose $F = G^{-1}$ then $\nabla F(\underline{x}) = (\nabla G(\underline{X}(\underline{x})))^{-1}$ and

$$\forall\, \underline{x} \in G(\mathcal{U}) : \nabla F(\underline{x})\underline{v}(\underline{x}) = \underline{e_1} . \tag{6}$$

Inserting now $\underline{x} = G(\underline{X})$ in $\dot{\underline{x}} = \underline{v}(\underline{x})$ we obtain ($\dot{} = d/dt$)

$$\forall\, \underline{X} \in \mathcal{U} : \nabla G(\underline{X})\dot{\underline{X}} = \underline{v}(G(\underline{X})) = \nabla G(\underline{X})\underline{e_1} .$$

hence

$$\boxed{\forall\, \underline{X} \in \mathcal{U} : \dot{\underline{X}} = \underline{e_1}} .$$

□

Example 1.10 and 1.11
(a) Find the flux integral of the differential equation $y' = e^y \sin x$. For which $y_0 \in \mathbb{R}$ does the solution of the initial value problem

$$y' = e^y \sin x, \quad y(0) = y_0$$

"globally" exist, i.e. on the entire line $\mathbb{R}$?
(b) Find the flux integral of the differential equation

$$y' = (x - y + 3)^2$$

and sketch the general solution (by means of isoklines).
Hint: Use the substitution $z = x - y + 3$ and consider the subdomains
$|z| < 1,\ |z| = 1,\ |z| > 1.$

Solution: (a) Separation of variables yields

$$\int_{y_0}^{y} e^{-y}\, dy = \int_{x_0}^{x} \sin x \, dx ,$$

hence

$$e^{-y} - e^{-y_0} = \cos x - \cos x_0 \Longrightarrow \Phi(x; x_0, y_0) = -\ln(\cos x + \exp(-y_0) - \cos x_0)\,.$$

Since the argument of the logarithmus must be positiv there exists a global solution for

$$e^{-y_0} - \cos x_0 > 1 \Longleftrightarrow e^{-y_0} > \cos x_0 + 1\,.$$

Because of the monotony of the logarithmus then

$$-y_0 > \ln(\cos x_0 + 1) \Longleftrightarrow y_0 < -\ln(\cos x_0 + 1) \Longrightarrow y(0) < -\ln 2\,.$$

(b) For $x - y + 3 = z$ and $z = \pm 1$ we obtain the solutions

$$y = x + 2, \qquad y = x + 4\,,$$

resp. $\Phi(x, x_0, y_0) = y_0 + x - 2 - (x_0 - 2)$ etc..
Solutions cannot intersect because y' is given explicitely. They must therefore remain for $|z| \neq 1$ entirely in one of the following domains:

$$G_1 = \{(x, y),\ |x-y+3| < 1\}, \quad G_2 = \{(x, y),\ x-y+3 < -1\}, \quad G_3 = \{(x, y),\ x-y+3 > 1\}.$$

For z we obtain the differential equation

$$z' = 1 - y' = 1 - (x - y + 3)^2 = 1 - z^2\,.$$

Separation of variables yields

$$\frac{z'}{1 - z^2} = 1$$

By partial decomposition

$$\frac{z'}{1 - z} + \frac{z'}{1 + z} = 2\,.$$

Integration yields

$$-\ln|z - 1| + \ln|z + 1| = 2x + c_1 \quad \Longrightarrow \quad \frac{|z + 1|}{|z - 1|} = ce^{2x}$$

Fallunterscheidung:

$$\begin{aligned}
z > 1: &\quad \frac{z + 1}{z - 1} = ce^{2x} &&\Longrightarrow \quad z = \frac{ce^{2x} + 1}{ce^{2x} - 1}\\
|z| < 1: &\quad \frac{z + 1}{-(z - 1)} = ce^{2x} &&\Longrightarrow \quad z = \frac{ce^{2x} - 1}{ce^{2x} + 1},\\
z < -1: &\quad \frac{-(z + 1)}{-(z - 1)} = ce^{2x} &&\Longrightarrow \quad z = \frac{ce^{2x} + 1}{ce^{2x} - 1}
\end{aligned}$$

Result:

$$\boxed{y = x + 3 - \frac{ce^{2x} + 1}{ce^{2x} - 1}, \quad (x, y) \in G_2 \cup G_3, \qquad y = x + 3 - \frac{ce^{2x} - 1}{ce^{2x} + 1}, \quad (x, y) \in G_1}\,.$$

For the calculation of the flux integral, we insert $x = x_0$ and $y = y_0$ and solve w.r.t. the constant c, then the result for c is inserted. For the general initial value problem

$$y' = (x - y + 3)^2, \quad y(x_0) = y_0,$$

we obtain in $G_1 \cup G_2 \cup G_3$

$$\begin{aligned} y &= x + 3 - \frac{(x_0 - y_0 + 4)e^{2x} + (x_0 - y_0 + 2)e^{2x_0}}{(x_0 - y_0 + 4)e^{2x} - (x_0 - y_0 + 2)e^{2x_0}}, \quad (x_0, y_0) \in G_2 \cup G_3\,, \\ y &= x + 3 - \frac{(x_0 - y_0 + 4)e^{2x} - (x_0 - y_0 + 2)e^{2x_0}}{(x_0 - y_0 + 4)e^{2x} + (x_0 - y_0 + 2)e^{2x_0}}, \quad (x_0, y_0) \in G_1\,. \end{aligned}$$

To Section 1.7

Lemma 2 *(Lemma 1.12)*
(1°) *$\mathcal{H} \subset \mathcal{X}$ is a hyperplane if and only if*

$$\exists\, 0 \neq f : \mathcal{X} \to \mathbb{R} \text{ linear } \exists\, c \in \mathbb{R} : \mathcal{H} = \{x \in \mathcal{X},\ f(x) = c\}\,.$$

(2°) *If $0 \notin \mathcal{H} \subset \mathcal{X}$ is a hyperplane then there exists uniquely a linear functional $f : \mathcal{X} \to \mathbb{R}$ with $\mathcal{H} = \{x \in \mathcal{X}\,,\ f(x) = 1\}$.*
(3°) *If $0 \neq f : X \to \mathbb{R}$ is linear and $\mathcal{H} = \{x \in \mathcal{X},\ f(x) = c\}$ is a hyperplane then $\mathcal{H} = \overline{\mathcal{H}}$ closed if and only if the mapping f is continuous.*

Proof. Proof. Cf. [Luenberger69], p. 129.
(a) A hyperplane $\mathcal{H}$ is an affin subspace hence $\mathcal{H} = v + \mathcal{U}$ where $\mathcal{U}$ is a linear subspace and $v \in \mathcal{H}$.
Case 1. $v \notin \mathcal{U}$ then $2v \notin \mathcal{H}$ otherwise

$$\mathcal{H} \ni x = 2v \Longrightarrow x - v = v \in \mathcal{U}\,.$$

Hence $\mathcal{X} = \text{span}\{v, \mathcal{H}\} = \text{span}\{v, \mathcal{U}\}$ because $\mathcal{H}$ is maximum. Let $\mathcal{X} \ni x = \alpha v + u,\ u \in \mathcal{U}$, and $f(x) := \alpha$. Then f is linear and

$$\mathcal{H} = \{x \in X,\ f(x) = 1\}.$$

Case 2. $v \in \mathcal{U}$, then choose $w \notin \mathcal{U} \neq \mathcal{X}$. Then $\mathcal{X} = \text{span}\{w, \mathcal{U}\},\ \mathcal{H} = \mathcal{U}$. Let $\mathcal{X} \ni x = \alpha w + u,\ u \in \mathcal{U}$ and $f(x) = \alpha$. Then f is linear and

$$\mathcal{H} = \{x \in \mathcal{X},\ f(x) = 0\}.$$

(b) Let $0 \neq f : \mathcal{X} \to \mathbb{R}$ be linear and $\mathcal{U} = \{x \in \mathcal{X},\ f(x) = 0\}$. Then $\mathcal{U}$ is a subspace. Let $v \in \mathcal{X}$ with $f(v) = 1$ which exists because we obtain $f = 0$ from $f(x) = 0$ for all $x \in \mathcal{X}$. Then $v \notin \mathcal{U}$ and it follows that

$$\forall\, x \in \mathcal{X} :\ f(x - f(x)v) = f(x) - f(x)f(v) = f(x) - f(x) = 0.$$

By this way it follows that $x - f(x)v \in \mathcal{U}$ hence $\mathcal{X} = \text{span}\{v, \mathcal{U}\}$. Because $v \notin \mathcal{U}$ we find that $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{U} \neq \mathcal{X}$ hence $\mathcal{U}$ must be a hyperplane.
For arbitrary $c \in \mathbb{R}$ let $w \in \mathcal{X}$ with $f(w) = c$. Then $w \notin \mathcal{U}$ and

$$\begin{aligned} &\{x \in X,\ f(x) = c\} = \{x \in X,\ f(x - w) = 0\} \\ &= \{x \in X,\ x - w \in U\} = w + U \end{aligned}$$

is a hyperplane.
(2°) The existence follows from (1°). Let $f, g : \mathcal{X} \to \mathbb{R}$ be linear with

$$\mathcal{H} = \{x \in \mathcal{X},\ f(x) = 1\} = \{x \in X,\ g(x) = 1\}$$

then $\mathcal{H} \subset \mathcal{W} := \{x \in \mathcal{X},\ f(x) - g(x) = 0\}$. Choose $x \in \mathcal{H}$ then $-x \notin \mathcal{H}$ because else $0 \in \mathcal{H}$. But $-x \in \mathcal{W}$ hence

$$\mathcal{W} = \mathrm{span}\{x, \mathcal{H}\} = \mathcal{X}.$$

By this way we obtain $f(x) = g(x)$ for all $x \in \mathcal{X}$ hence $f = g$.
(3°) Cf. [Luenberger 69], p. 130.□
If $\mathcal{H}$ is a hyperplane then either $\mathcal{H}$ is closed, i.e., $\mathcal{H} = \overline{\mathcal{H}}$ or $\mathcal{H}$ is dense in $\mathcal{X}$, i.e. $\overline{\mathcal{H}} = \mathcal{X}$; cf. [Taylor] p. 139. For example, $\mathcal{C}[0,1]$ is dense in $\mathcal{L}^2[0,1]$ hence a hyperplane; cf. [Pflaumann-Unger], p. 187.