On the solvability of some systems of integro-differential equations with anomalous diffusion in higher dimensions

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Pure and Applied Functional Analysis

To the memory of Louis Nirenberg

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Existence of stationary solutions of nonlocal reaction- diffusion equations: existence of biological species.

1. Introduction

The integro-differential systems of equations: nonlocal consumption of resources, intra-specific competition. Nonlocal interaction of neurons.

$$\frac{\partial u_m}{\partial t} = -D_m(-\Delta)^{s_m} u_m + \int_{\mathbb{R}^d} K_m(x-y) g_m(u(y,t))dy + f_m(x), \qquad (1)$$

$$d = 4, 5, \quad 1 \leq m \leq N, \quad \frac{3}{2} - \frac{d}{4} < s_m < 1$$

from cell population dynamics. Cell genotype is x, cell density distributions for various groups of cells as functions of their genotype and time are $u_m(x,t)$,

$$u(x,t) = (u_1(x,t), u_2(x,t), ..., u_N(x,t))^T.$$

The evolution of cell density is due to cell proliferation, mutations and cell influx/efflux. The change of genotype due to small random mutations-anomalous diffusion terms. Large mutations are via the integral production terms. $g_m(u)$ are the rates of cell birth, depend on u

(density dependent proliferation). $K_m(x-y)$ are the proportions of newly born cells changing their genotype from y to x, depend on the distance between the genotypes. $f_m(x)$ are the influxes/effluxes of cells for different genotypes. Proved the existence of a stationary solution in $H^3(\mathbb{R}^d, \mathbb{R}^N)$. The space variable corresponds to cell genotype.

Anomalous diffusion problem with $(-\Delta)^s$: defined via the spectral calculus, namely for $d = 4, 5$

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \widehat{f}(p) e^{ipx} dp, \quad (-\Delta)^s f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} |p|^{2s} \widehat{f}(p) e^{ipx} dp.$$

$d > 5$: higher order Sobolev spaces for the Sobolev embedding.

Standard Laplacian: V.V., V. Volpert, Disc., Nonlin., and Complex., (2016) in $\mathbb{R}^5$.

Anomalous diffusion: plasma physics and turbulence.

B.Carreras, V.Lynch, G.Zaslavsky, Phys. Plasmas (2001).

Surface diffusion.

J.Sancho, A. Lacasta, K.Lindenberg, I.Sokolov, A.Romero, Phys. Rev. Lett. (2004).

Semiconductors.

H.Scher, E.Montroll, Phys. Rev. B (1975).

Physical meaning: the random process occurs with longer jumps in comparison with normal diffusion.

Normal diffusion: finite moments of jump length distribution.

Anomalous diffusion: not the case.

R. Metzler, J. Klafter, Phys. Rep. (2000).

The existence of stationary solutions, Pure and Applied Functional Analysis. To the memory of Louis Nirenberg.

For $d = 4, 5,\ 1 \le m \le N,\ \dfrac{3}{2} - \dfrac{d}{4} < s_m < 1$

$$-(-\Delta)^{s_m} u_m + \int_{\mathbb{R}^d} K_m(x-y) g_m(u(y)) dy + f_m(x) = 0. \qquad (2)$$

For $d = 4 : \dfrac{1}{2} < s_m < 1$ and for $d = 5 : \dfrac{1}{4} < s_m < 1$.

Set $D_m = 1,\ K_m(x) = \varepsilon_m \mathcal{K}_m(x),\ \varepsilon_m \ge 0$ small parameter and $\varepsilon = \max_{1 \le m \le N} \varepsilon_m$.

In one dimension, V.V., V.Volpert, Springer (2017), $0 < s_m = s < \dfrac{1}{4}$.

In two dimensions, V.V., Pure Appl. Funct. Anal. (2020), $0 < s_m < \dfrac{1}{2}$.

In $\mathbb{R}^3$, V.V., V.Volpert, Disc., Nonlin.. and Complex., (2016), $\dfrac{1}{4} < s_m = s < \dfrac{3}{4}$. In $d = 3, 4, 5$ Sobolev inequality for the fractional negative Laplacian.

E.H. Lieb, Ann. of Math., (1983). For $d = 4, 5,\ 1 \le m \le N$

$$\|f_m(x)\|_{L^{\frac{2d}{d-6+4s_m}}(\mathbb{R}^d)} \le c_{sob}\|(-\Delta)^{\frac{3}{2}-s_m} f_m(x)\|_{L^2(\mathbb{R}^d)}, \quad \frac{3}{2} - \frac{d}{4} < s_m < 1.$$

Hence

$$f_m(x) \in L^1(\mathbb{R}^d) \cap L^{\frac{2d}{d-6+4s_m}}(\mathbb{R}^d).$$

For

$$d = 4 : f_m(x) \in L^1(\mathbb{R}^4) \cap L^{\frac{4}{2s_m-1}}(\mathbb{R}^4),$$

for

$$d = 5 : f(x) \in L^1(\mathbb{R}^5) \cap L^{\frac{10}{4s_m-1}}(\mathbb{R}^5).$$

Standard interpolation argument:

$f_m(x) \in L^2(\mathbb{R}^d), \quad d = 4, 5, \quad 1 \le m \le N.$

Fractional Sobolev norm

$$\|\phi\|^2_{H^{2s}(\mathbb{R}^d)} := \|\phi\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^s\phi\|^2_{L^2(\mathbb{R}^d)}, \quad 0 < s \le 1.$$

Standard Sobolev embedding

$$\|\phi\|_{L^\infty(\mathbb{R}^d)} \le c_e \|\phi\|_{H^3(\mathbb{R}^d)}, \quad c_e > 0, \quad d = 4, 5.$$

Here c_e is the constant of the embedding. The vector function's norm

$$\|u\|^2_{H^3(\mathbb{R}^d, \mathbb{R}^N)} := \sum_{m=1}^N \|u_m\|^2_{H^3(\mathbb{R}^d)} = \sum_{m=1}^N \{\|u_m\|^2_{L^2(\mathbb{R}^d)} + \|(-\Delta)^{\frac{3}{2}} u_m\|^2_{L^2(\mathbb{R}^d)}\}.$$

When all the parameters ε_m vanish, we obtain the generalized Poisson type equations

$$(-\Delta)^{s_m} u_m(x) = f_m(x), \quad x \in \mathbb{R}^d, \quad d = 4, 5, \quad 1 \le m \le N. \tag{3}$$

The standard Fourier transform

$$\widehat{\phi}(p) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \phi(x) e^{-ipx} dx, \quad d = 4, 5. \tag{4}$$

Upper bound

$$\|\widehat{\phi}(p)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{(2\pi)^{\frac{d}{2}}}\|\phi(x)\|_{L^1(\mathbb{R}^d)}, \quad d=4,5. \tag{5}$$

2. Solvability conditions for the generalized Poisson equation.

$$(-\Delta)^s u = f(x), \quad x \in \mathbb{R}^d, \quad d=4,5, \quad 0<s<1. \tag{6}$$

Let $u(x) \in L^2(\mathbb{R}^d)$, assume $f(x) \in L^2(\mathbb{R}^d)$. Then $u(x) \in H^{2s}(\mathbb{R}^d)$ as well.

Uniqueness.

Let $u_1(x),\ u_2(x) \in H^{2s}(\mathbb{R}^d)$ solve (6). $w(x) = u_1(x) - u_2(x) \in H^{2s}(\mathbb{R}^d)$.

$$(-\Delta)^s u_1 = f(x), \quad (-\Delta)^s u_2 = f(x).$$

Hence

$$(-\Delta)^s w = 0.$$

$(-\Delta)^s : H^{2s}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ no nontrivial zero modes. Then w vanishes in

$\mathbb{R}^d$. Apply Fourier transform (4) to (6).

$$\widehat{u}(p) = \frac{\widehat{f}(p)}{|p|^{2s}} = \frac{\widehat{f}(p)}{|p|^{2s}} \chi_{\{|p| \le 1\}} + \frac{\widehat{f}(p)}{|p|^{2s}} \chi_{\{|p| > 1\}}.$$

Second term

$$\left| \frac{\widehat{f}(p)}{|p|^{2s}} \chi_{\{|p| > 1\}} \right| \le |\widehat{f}(p)| \in L^2(\mathbb{R}^d).$$

First term via (5)

$$\left\| \frac{\widehat{f}(p)}{|p|^{2s}} \chi_{\{|p| \le 1\}} \right\|^2_{L^2(\mathbb{R}^d)} \le \frac{\|f\|^2_{L^1(\mathbb{R}^d)}}{(2\pi)^d} \frac{|S^d|}{d - 4s} < \infty.$$

No orthogonality conditions are required for right side $f(x)$.

In $\mathbb{R}^3$ when $\frac{3}{4} \le s < 1$ and additionally $|x|f(x) \in L^1(\mathbb{R}^3)$, equation (6) has a unique solution $u(x) \in H^{2s}(\mathbb{R}^3)$ if and only if

$$\int_{\mathbb{R}^3} f(x)dx = (f(x), 1)_{L^2(\mathbb{R}^3)} = 0. \tag{7}$$

Note that $\widehat{f}(0) = \dfrac{1}{(2\pi)^{\frac{3}{2}}} \displaystyle\int_{\mathbb{R}^3} f(x)dx = 0.$

Unique solution of each equation (3) with $1 \le m \le N$

$$u_{0,m}(x) \in H^{2s_m}(\mathbb{R}^d), \quad \frac{3}{2} - \frac{d}{4} < s_m < 1, \quad d = 4, 5.$$

The space variable corresponds to cell genotype, not the usual physical space. The space dimension is not limited to $d = 4, 5$.

$$(-\Delta)^{s_m} u_{0,m} = f_m(x).$$

Then

$$(-\Delta)^{\frac{3}{2}} u_{0,m}(x) = (-\Delta)^{\frac{3}{2} - s_m} f_m(x) \in L^2(\mathbb{R}^d), \quad 1 \le m \le N$$

as assumed. Hence

$$u_{0,m}(x) \in H^3(\mathbb{R}^d).$$

and

$$u_0(x) := (u_{0,1}(x), u_{0,2}(x), ..., u_{0,N}(x))^T \in H^3(\mathbb{R}^d, \mathbb{R}^N).$$

3. Fixed point argument

Seek the resulting solution of the stationary nonlinear problem (2) as

$$u(x) = u_0(x) + u_p(x), \tag{8}$$

$$u_p(x) := (u_{p,1}(x), u_{p,2}(x), ..., u_{p,N}(x))^T.$$

Perturbative system of equations, with
$\frac{3}{2} - \frac{d}{4} < s_m < 1, \quad 1 \le m \le N, \quad d = 4, 5$

$$(-\Delta)^{s_m} u_{p,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} \mathcal{K}_m(x-y) g_m(u_0(y) + u_p(y)) dy. \tag{9}$$

The Fixed Point argument in a closed ball in the Sobolev space:

$$B_\rho = \{u(x) \in H^3(\mathbb{R}^d, \mathbb{R}^N) \mid \|u\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \leq \rho\}, \quad 0 < \rho \leq 1. \tag{10}$$

Seek the solution of (9) as the fixed point of the auxiliary nonlinear problem with $\frac{3}{2} - \frac{d}{4} < s_m < 1, \ 1 \leq m \leq N, \ d = 4, 5$

$$(-\Delta)^{s_m} u_m(x) = \varepsilon_m \int_{\mathbb{R}^d} \mathcal{K}_m(x-y) g_m(u_0(y) + v(y)) dy \tag{11}$$

in ball (10). Non Fredholm operators

$$(-\Delta)^{s_m} : H^{2s_m}(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

The essential spectrum $\sigma_{ess}((-\Delta)^{s_m}) = [0, +\infty)$, no bounded inverse.

V.V., V.Volpert, Doc. Math. (2011),

V.V., V.Volpert, Anal. Math. Phys. (2012)

relied on the orthogonality relations.

V.V., Math. Model. Nat. Phenom., (2010).

The fixed point technique to estimate the perturbation to the standing solitary wave

$$\psi(x,t) = \phi(x)e^{i\omega t}$$

of the Nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\psi + F(|\psi|^2)\psi$$

when small perturbation is applied either to the potential or to the nonlinear term. The Schrödinger operator involved had the Fredholm property.

When $K_m(x) = \delta(x),\ 1 \le m \le N$ is Dirac's delta measure, the system of standard nonlinear heat equations.

The existence of stationary solutions in the case of the standard Laplacians in $H^3(\mathbb{R}^5)$.

V.V., V. Volpert, Disc., Nonlin., and Complex. (2016).

The operator T_g via the auxiliary nonlinear problem (11), such that $u = T_g v, \quad u$ is a solution. Our main result is as follows.

Theorem 1. Under our technical assumptions problem (11) defines the map $T_g : B_\rho \to B_\rho$, which is a strict contraction for all $0 < \varepsilon \leq \varepsilon^*$ for a certain $\varepsilon^* > 0$. The unique fixed point $u_p(x)$ of the map T_g is the only solution of problem (9) in B_ρ.

The resulting stationary solution of (2) is nontrivial: the source terms $f_m(x)$ are nontrivial for some $1 \leq m \leq N$ and all $g_m(0) = 0$ as assumed.

Proof. Choose arbitrarily $v(x) \in B_\rho$, denote $G_m(x) := g_m(u_0(x) + v(x))$. Apply the standard Fourier transform (4) to (11). Thus

$$\widehat{u}_m(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}_m(p) \widehat{G}_m(p)}{|p|^{2s_m}}, \quad 1 \leq m \leq N, \quad d = 4, 5.$$

The norm with d=4,5

$$\|u_m\|^2_{L^2(\mathbb{R}^d)} = (2\pi)^d \varepsilon_m^2 \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G}_m(p)|^2}{|p|^{4s_m}} dp, \quad 1 \le m \le N.$$

Express $\int_{\mathbb{R}^d} dp = \int_{|p|\le R} dp + \int_{|p|>R} dp$ with $R \in (0, +\infty)$ and minimize over R. We derive

$$\|u\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \le \varepsilon C \le \rho$$

for all $\varepsilon > 0$ small enough, such that $u(x) \in B_\rho$ as well.

Uniqueness.

If for some $v(x) \in B_\rho$ there are two solutions $u_{1,2}(x) \in B_\rho$ of (11), their difference $w(x) := u_1(x) - u_2(x) \in L^2(\mathbb{R}^d, \mathbb{R}^N)$ solves

$$(-\Delta)^{s_m} w_m(x) = 0, \quad \frac{3}{2} - \frac{d}{4} < s_m < 1, \quad 1 \le m \le N.$$

Each $(-\Delta)^{s_m}$ in $\mathbb{R}^d$: no nontrivial square integrable zero modes, $w(x)$ vanishes in $\mathbb{R}^d$.

Then (11) defines a map $T_g : B_\rho \to B_\rho$ for all $\varepsilon > 0$ small enough.

To show that this map is a strict contraction.

Choose arbitrarily $v_{1,2}(x) \in B_\rho$. Then $u_{1,2} := T_g v_{1,2} \in B_\rho$ as well, ε small enough. For $\frac{3}{2} - \frac{d}{4} < s_m < 1,\ 1 \le m \le N,\ d = 4, 5$

$$(-\Delta)^{s_m} u_{1,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} \mathcal{K}_m(x-y) g_m(u_0(y) + v_1(y))dy,$$

$$(-\Delta)^{s_m} u_{2,m}(x) = \varepsilon_m \int_{\mathbb{R}^d} \mathcal{K}_m(x-y) g_m(u_0(y) + v_2(y))dy.$$

Introduce $G_{1,m}(x) := g_m(u_0(x) + v_1(x)),\ G_{2,m}(x) := g_m(u_0(x) + v_2(x))$. Apply the standard Fourier transform (4). Arrive at

$$\widehat{u}_{1,m}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}_m(p)\widehat{G_{1,m}}(p)}{|p|^{2s_m}}, \quad \widehat{u}_{2,m}(p) = \varepsilon_m (2\pi)^{\frac{d}{2}} \frac{\widehat{\mathcal{K}}_m(p)\widehat{G_{2,m}}(p)}{|p|^{2s_m}}.$$

Write the norm

$$\|u_{1,m} - u_{2,m}\|^2_{L^2(\mathbb{R}^d)} = \varepsilon_m^2 (2\pi)^d \int_{\mathbb{R}^d} \frac{|\widehat{\mathcal{K}}_m(p)|^2 |\widehat{G_{1,m}}(p) - \widehat{G_{2,m}}(p)|^2}{|p|^{4s_m}} dp.$$

Express

$$\int_{\mathbb{R}^d} dp = \int_{|p|\leq R} dp + \int_{|p|>R} dp,$$

minimize over $R \in (0, +\infty)$. Estimate the norm

$$\|u_1 - u_2\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)} \leq \varepsilon C \|v_1 - v_2\|_{H^3(\mathbb{R}^d, \mathbb{R}^N)}.$$

The map $T_g : B_\rho \to B_\rho$ defined by (11) is a strict contraction for all $\varepsilon > 0$ small enough. Unique fixed point $u_p(x)$, the only solution of the perturbative system of equations (9) in B_ρ.

The resulting solution of the stationary problem (2):

$$u(x) = u_0(x) + u_p(x) \in H^3(\mathbb{R}^d, \mathbb{R}^N),$$

where $u_{0,m}(x),\ 1 \leq m \leq N$ solves the generalized Poisson equation (3).

Also proved: cumulative $u(x)$ is continuous in $H^3(\mathbb{R}^d, \mathbb{R}^N)$ with respect to the nonlinear, twice continuously differentiable vector function $g(z)$.

4. Discussion of the future work.

1. To study the convergence of the solutions $u(x,t)$ of problem (1) to the equilibrium.

2. To generalize the results above to the case when the normal diffusion is combined with the anomalous diffusion in a single integro-differential equation or a system of coupled integro-differential equations. M.Efendiev, V.V., J.Differential Equations (2021).

3. To perform the iterations of the kernels of an integro-differential equation and to show the existence of its stationary solution in the sense of sequences.

4. To work on the preservation of the nonnegativity of solutions of the systems of parabolic equations. M.Efendiev, V.V., Springer chapters (2021).