Time-frequency transforms in Euclidean spaces

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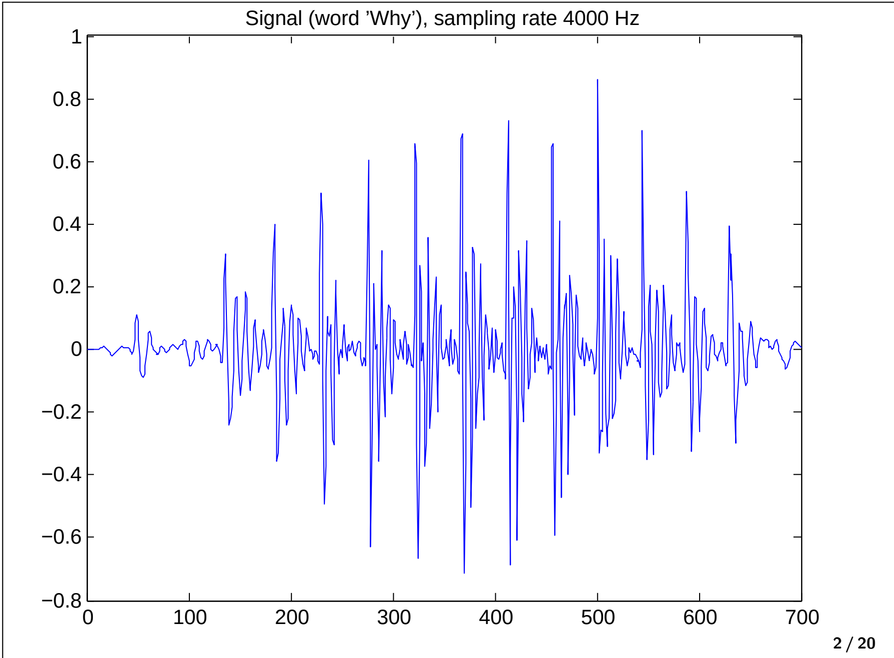
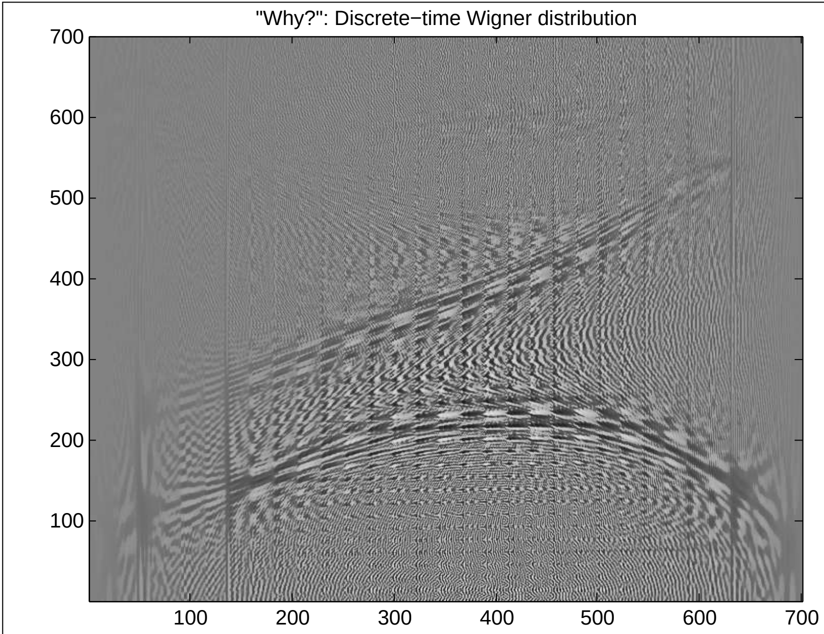
Abstract: Time-frequency analysis can be described as Fourier analysis simultaneously both in time and in frequency. A time-frequency transform is a sesquilinear mapping from a family of test functions in a Euclidean space to functions in the time-frequency plane. The class of time-frequency transforms is further restricted by imposing conditions stemming from basic transformations of signals and those which an idealized energy density could satisfy. We characterize time-frequency transforms in terms of the corresponding pseudo-differential operator quantizations and integral kernel conditions.

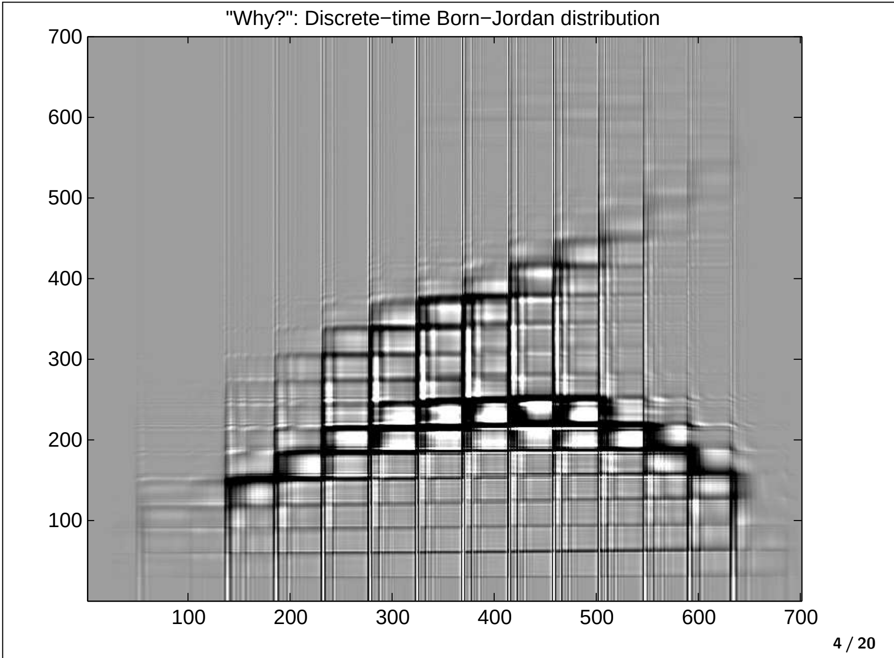
Waveform of speech...



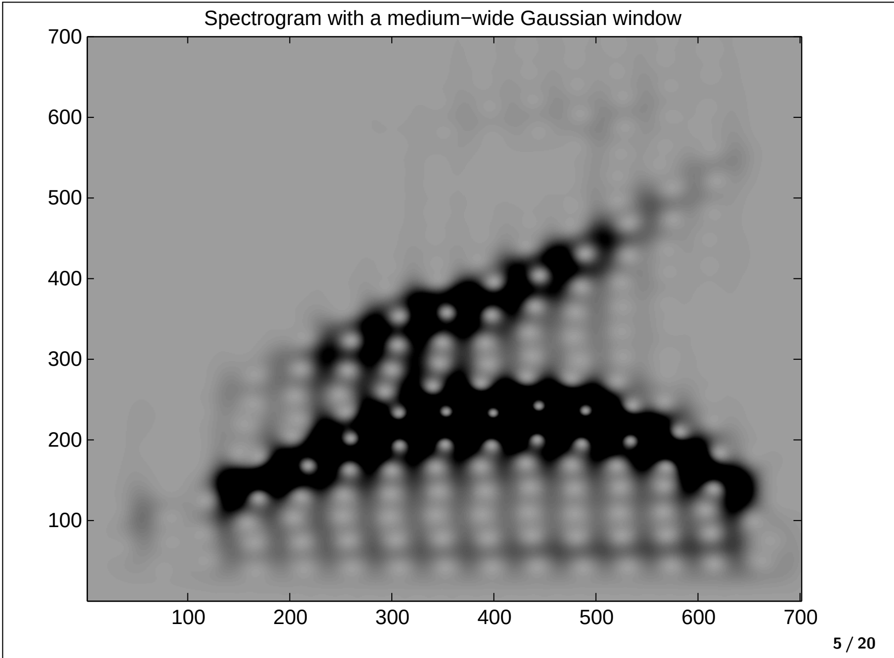
Wigner for speech...



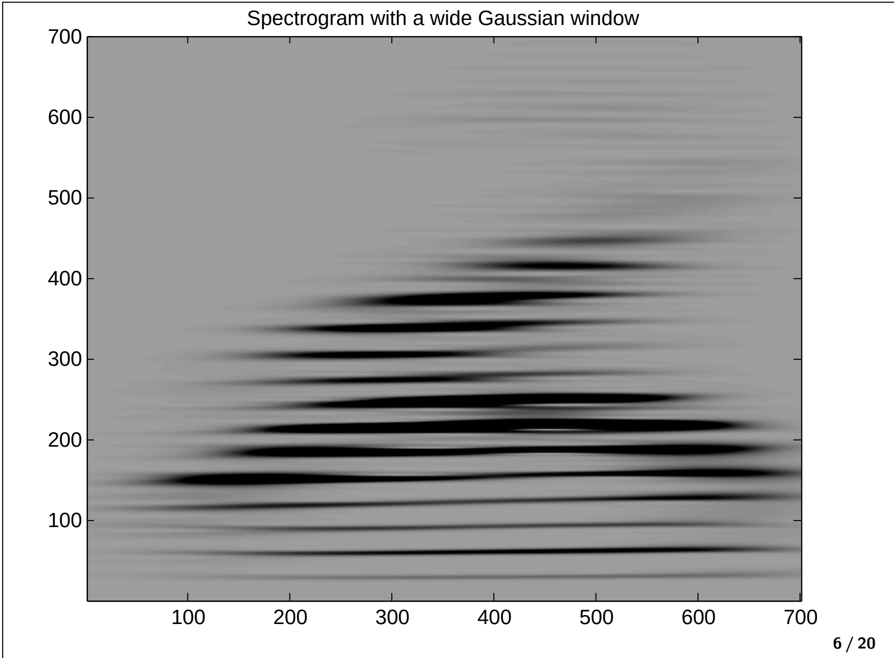
... Born–Jordan for speech...



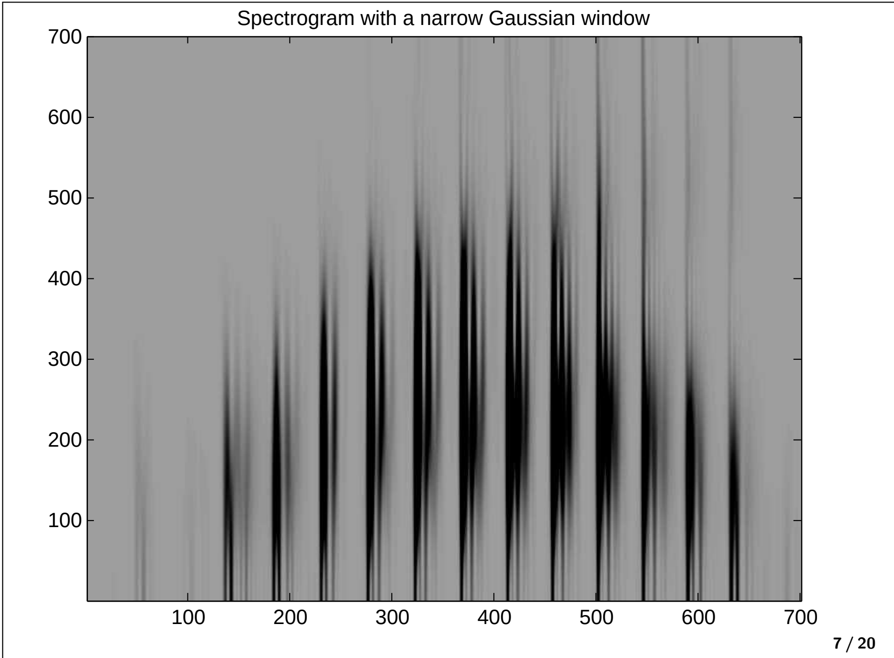
... a spectrogram with a Gaussian window...



... a spectrogram with wide Gaussian window...



... a spectrogram with narrow Gaussian window



Time-frequency transforms $Q = \psi * W$ [Cohen 1966]

Informal idea:

$$Q[u](x,\eta) := Q(u,u)(x,\eta)$$

is the "energy density" ("time-frequency distribution") of signal $u : \mathbb{R}^n \to \mathbb{C}$ at time-frequency $(x,\eta) \in \mathbb{R}^n \times \widehat{\mathbb{R}}^n$, where $\widehat{\mathbb{R}} \cong \mathbb{R}$.

For signals $u, v : \mathbb{R}^n \to \mathbb{C}$, a time-frequency transform $Q(u,v) : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \to \mathbb{C}$ is a time-frequency shift-invariant family of sesquilinear forms:

$$Q(M_\xi T_y u, M_\xi T_y v)(x,\eta) = Q(u,v)(x-y, \eta-\xi),$$

for all the translation-modulations

$$M_\xi T_y u(x) := \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\, u(x-y)$$

(*time-lag* $y \in \mathbb{R}^n$, *frequency-lag* $\xi \in \widehat{\mathbb{R}}^n$). In practice, then

$$Q(u,v) = \psi * W(u,v),$$

where $W(u,v) : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \to \mathbb{C}$ is the Wigner transform.

Q-quantization (Q-pseudo-differential operators)

Fix a time-frequency transform $Q = \psi * W$ with smooth $\widehat{\psi} \in L^\infty$ having polynomially bounded derivatives. The corresponding Q-quantization is the mapping

$$a \mapsto a^Q = \mathrm{Op}_Q(a) \qquad (1)$$

defined by the L^2-inner products

$$\langle u, a^Q v \rangle = \langle Q(u,v), a \rangle, \quad \text{or equivalently} \qquad (2)$$

$$\langle v, a^Q v \rangle = \langle Q(v,v), a \rangle. \qquad (3)$$

Idea: Operator a^Q sees **only** the energy density $Q[v] = Q(v,v)$.
$u : \mathbb{R}^n \to \mathbb{C}$ is a test signal,
$v : \mathbb{R}^n \to \mathbb{C}$ is an input signal,
$a^Q v : \mathbb{R}^n \to \mathbb{C}$ the corresponding output signal,
$a : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \to \mathbb{C}$ a symbol (or weight) of the pseudo-differential operator a^Q (time-varying filter). Distributional symbols are ok: important case $Q_0 = a^Q$ for the Dirac delta symbol $a = \delta = \delta_{(0,0)}$.
The properties of time-frequency transform Q and the Q-quantization $a \mapsto a^Q$ are naturally related.

Wigner transform (1931), Weyl quantization (1927)

Signals $u, v : \mathbb{R}^n \to \mathbb{C}$ have the *Wigner transform*
$W(u,v) : \mathbb{R}^n \times \widehat{\mathbb{R}}^n \to \mathbb{C}$,

$$W(u,v)(x,\eta) := \int \mathrm{e}^{-\mathrm{i}2\pi y\cdot\eta}\, u(x+y/2)\, v(x-y/2)^*\, \mathrm{d}y. \tag{4}$$

W is a symmetric time-frequency transform.
The Wigner distribution $[v] \mapsto W[v] = W(v,v)$ is invertible, has correct marginal energy densities, is scale-invariant, is time-local, and is frequency-local. **But:** Wigner transform is very sensitive to noise, and thus it is not useful in practise.
The corresponding *(Wigner-)Weyl quantization* $a \mapsto a^W$, where

$$a^W v(x) = \iint \mathrm{e}^{\mathrm{i}2\pi(x-y)\cdot\eta}\, a(\frac{x+y}{2},\eta)\, v(y)\, \mathrm{d}y\, \mathrm{d}\eta. \tag{5}$$

Warning! When trying to generalize the times $\mathbb{R}^n$ to groups G:
typically non-invertible mapping $(y \mapsto y+y) : G \to G$
(in the multiplicative notation, non-invertible $(y \mapsto y^2) : G \to G$).
Then there is no reasonable analogy to Wigner/Weyl on such G!
(But then just exploit the Kohn–Nirenberg quantization... ;)

Q time-frequency transform,
$Q_0 = \delta^Q$ its "evaluation at the origin",
ϕ_Q the ambiguity kernel,
$\mathrm{Op}_Q = (a \mapsto a^Q)$ corresponding pseudo-differential quantization.
Symplectic transform $T_{\mathcal{A}} a(x, \eta) := a(\mathcal{A}(x, \eta))$ for $\mathcal{A} \in \mathrm{Sp}(2n)$.

Theorem 3. (Symplectic covariance)

Let Q be a time-frequency transform, $\mathcal{A} \in \mathrm{Sp}(2n)$. Let $\mu(\mathcal{A})$ be a metaplectic operator such that $\Pi \circ \mu(\mathcal{A}) = \mathcal{A}$.
Then the following are equivalent:
(a) Q *is covariant under* $\mathcal{A}$.
(b) Q_0 *commutes with* $\mu(\mathcal{A})$.
(c) $\phi_Q = T_{\mathcal{A}^{-1}} \phi_Q$.
(d) $\mathrm{Op}_Q(T_{\mathcal{A}} a) = \mu(\mathcal{A})^{-1} \circ \mathrm{Op}_Q(a) \circ \mu(\mathcal{A})$.

Time-frequency transform Q is *symmetric* if $Q[u] = Q(u, u)$ is real-valued for all $u \in \mathscr{S}(\mathbb{R}^n)$.

Quantization $Op_Q = (a \mapsto a^Q)$ is *symmetric* if

$$(a^*)^Q = (a^Q)^*$$

for all $a \in \mathscr{S}(\mathbb{R}^n \times \widehat{\mathbb{R}}^n)$, i.e.

$$\langle a^Q u, v \rangle = \langle u, a^Q v \rangle$$

for all $u, v \in \mathscr{S}(\mathbb{R}^n)$.

Theorem 5.

The following conditions are equivalent:

(a) Q *is symmetric.*

(b) Q_0 *is symmetric.*

(c) ϕ_Q *satisfies* $\phi_Q(y, \xi)^* = \phi_Q(-y, -\xi)$.

(d) Op_Q *is symmetric.*

Positivity

Time-frequency transform Q is *positive* if $Q[u] \geq 0$ for all $u \in \mathscr{S}(\mathbb{R}^n)$.

Quantization $\mathrm{Op}_Q = (a \mapsto a^Q)$ is *positive* if

$$\langle u, a^Q u \rangle \geq 0$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $a \in \mathscr{S}(\mathbb{R}^n \times \widehat{\mathbb{R}}^n)$.

Theorem 6.

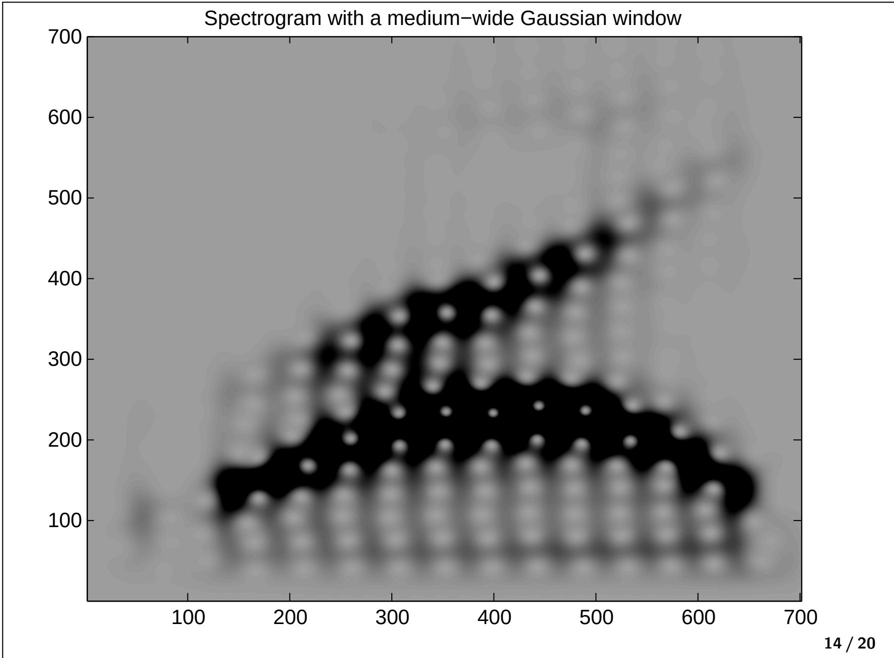
The following conditions are equivalent:

(a) Q *is positive.*

(b) Q_0 *is positive.*

(c) Op_Q *is positive.*

Spectrograms are positive



Normalization: correct traces

Time-frequency transform Q is *normalized* if

$$\iint Q[u](x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta = \|u\|^2$$

for all $u \in \mathscr{S}(\mathbb{R}^n)$.

Quantization $\mathrm{Op}_Q = (a \mapsto a^Q)$ has *correct traces* if

$$\mathrm{tr}(a^Q) = \iint a(x,\eta)\,\mathrm{d}x\,\mathrm{d}\eta$$

for all $a \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, where tr is the trace functional.

Theorem 7 (Normalization)

The following conditions are equivalent:

(a) Q *is normalized.*

(b) $\phi_Q(0,0) = 1$.

(c) Op_Q *has correct traces.*

Correct margins

Time-frequency transform Q has *correct frequency margins* if $\int Q[u](x,\eta)\,\mathrm{d}x = |\widehat{u}(\eta)|^2$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $\eta \in \widehat{\mathbb{R}}^n$.

Theorem 8 (Frequency margins)

The following conditions are equivalent:

(a) *Q has correct frequency margins.*

(b) $\phi_Q(y,0) = 1$.

(c) *If $a(x,\eta) = \widehat{f}(\eta)$ then $(\mathrm{Op}_Q a)v = f * v$ (i.e. convolution).*

Time-frequency transform Q has *correct time margins* if $\int Q[u](x,\eta)\,\mathrm{d}\eta = |u(x)|^2$ for all $u \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Theorem 9 (Time margins)

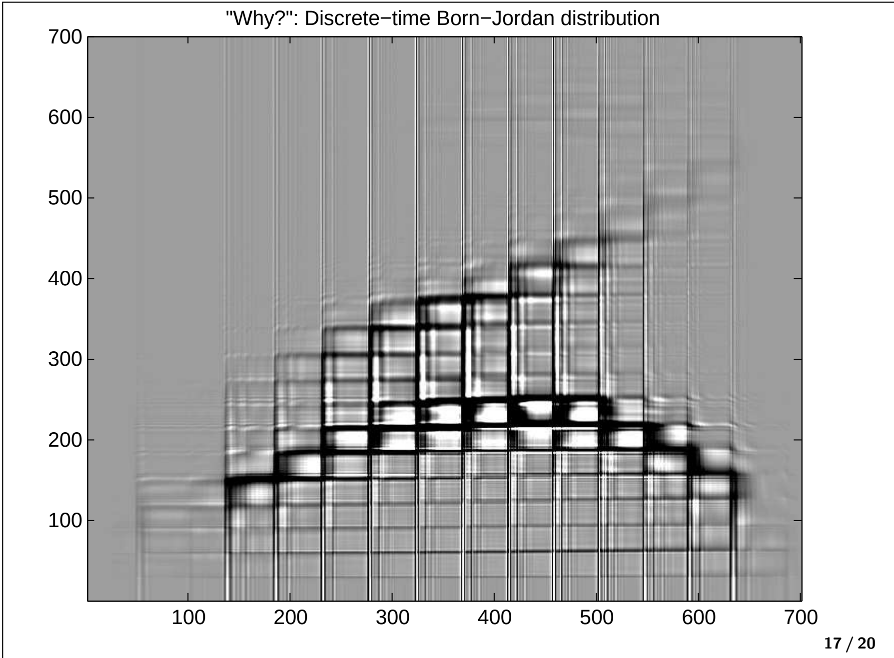
The following conditions are equivalent:

(a) *Q has correct time margins.*

(b) $\phi_Q(0,\xi) = 1$.

(c) *If $a(x,\eta) = g(x)$ then $(\mathrm{Op}_Q a)v = g\,v$ (i.e. multiplication).*

Born–Jordan has correct margins



Unitarity (Moyal property)

Time-frequency transform Q is *unitary* if it has the *Moyal property*

$$\langle Q(u,v), Q(f,g)\rangle = \langle u,f\rangle \langle v,g\rangle^* \tag{6}$$

for all $u, v, f, g \in \mathscr{S}(\mathbb{R}^n)$.

Quantization $\mathrm{Op}_Q = (a \mapsto a^Q)$ is *unitary* if

$$\langle a,b\rangle = \langle a^Q, b^Q\rangle \tag{7}$$

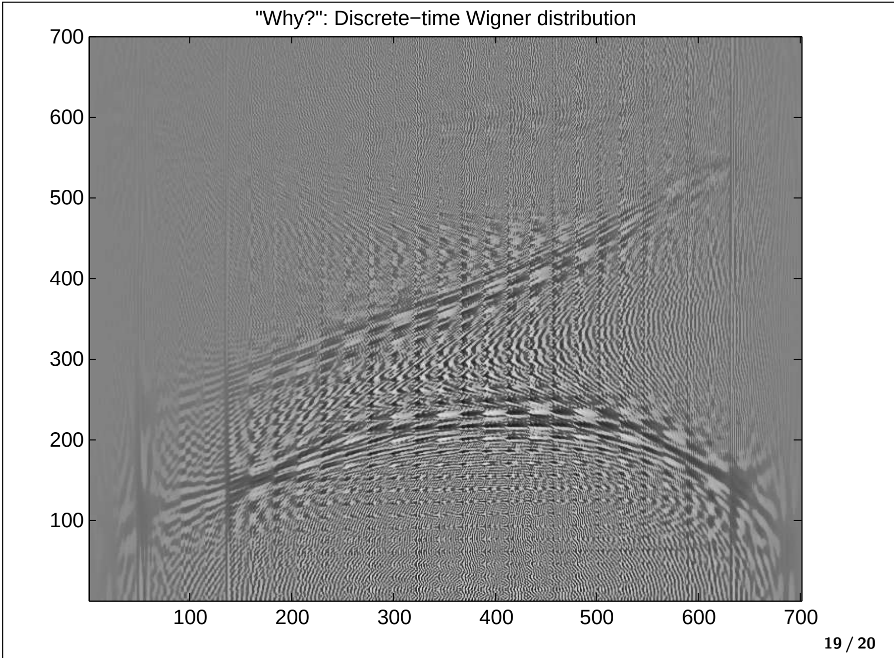
for all $a, b \in \mathscr{S}(\mathbb{R}^n \times \widehat{\mathbb{R}}^n)$.

Theorem 10 (Unitarity)

The following conditions are equivalent:
(a) Q *is unitary.*
(b) $|\phi_Q(\xi, y)| \equiv 1$.
(c) Op_Q *is unitary.*

Unitary transforms are sensitive to noise. Examples: Wigner and Kohn-Nirenberg are unitary.

Wigner is unitary



Born–Jordan is not unitary