Global dynamics for the critical Hardy-Sobolev parabolic equation below the ground state

Koichi Taniguchi (Tohoku Univ., AIMR)

The 13th International ISAAC Congress
Ghent, 2-6 August
Harmonic Analysis and Partial Differential Equations

Joint work with Noboru Chikami (Nagoya Inst. of Tech.)
and Masahiro Ikeda (RIKEN/Keio Univ.)

We study the Hardy-Sobolev parabolic equation:

$$\partial_t u - \Delta u = |x|^{-\gamma}|u|^{p-2}u, \quad t > 0,\ x \in \mathbb{R}^d,$$

where $d \geq 3$, $\gamma \in (0, 2)$ and $p > 2$.

Physical background

- The stationary problem $-\Delta u = |x|^{-\gamma}u^{p-1}$ has been proposed by Hénon as a model to study the *rotating stellar systems*.

Characteristics

- This equation has no classical solution and is not invariant under the translation w.r.t. x.

Hardy-Hénon parabolic equation

We consider the energy-critical Hardy-Sobolev parabolic equation:

$$\begin{cases} \partial_t u - \Delta u = |x|^{-\gamma}|u|^{2^*(\gamma)-2}u, & t > 0,\ x \in \mathbb{R}^d, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{HS}$$

where $d \geq 3$, $\gamma \in (0,2)$ and $2^*(\gamma) := \dfrac{2(d-\gamma)}{d-2}$ is the critical Hardy-Sobolev exponent.

Initial data

$$\|f\|_{\dot{H}^1} = \|f\|_{\dot{H}^1(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |\nabla f(x)|^2\,dx\right)^{\frac{1}{2}}.$$

Energy The energy functional $E : \dot{H}^1(\mathbb{R}^d) \to \mathbb{R}$ is given by

$$E(f) := \frac{1}{2}\|f\|_{\dot{H}^1}^2 - \frac{1}{2^*(\gamma)}\int_{\mathbb{R}^d} |x|^{-\gamma}|f(x)|^{2^*(\gamma)}\,dx.$$

Energy identity

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{R}^d} |\partial_t u(t,x)|^2 \, dx \le 0.$$

Scale invariance The equation (HS), $\dot{H}^1$-norm and the energy are invariant under

$$u_\lambda(t,x) := \lambda^{\frac{d-2}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Aim (Global dynamics below the ground state)

Our aim is to give a necessary and sufficient condition on initial data below the ground state W (i.e. $E(u_0) < E(W)$), under which the behavior of solutions to (HS) is completely dichotomized:

(i) $\|u_0\|_{\dot{H}^1} \le \|W\|_{\dot{H}^1} \Longrightarrow u$ is global & $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0$.

(ii) $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1} \Longrightarrow u$ blows up in finite or infinite time.

Definition of solution

Let $T > 0$ and $u_0 \in \dot{H}^1(\mathbb{R}^d)$.

A function $u = u(t,x)$ is called a (mild) solution to (HS) with $u(0) = u_0$ if

- $u \in C([0,T']; \dot{H}^1(\mathbb{R}^d)) \cap \mathcal{K}^q(T')$ for any $T' \in (0,T)$;
- u satisfies the integral equation

$$u(t,x) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-\tau)\Delta}\left\{ |\cdot|^{\gamma} |u(\tau,\cdot)|^{2^*(\gamma)-2} u(\tau,\cdot) \right\}(x)\, d\tau,$$

where $\{e^{t\Delta}\}_{t\ge 0}$ is the heat semigroup.

Auxiliary space

$$\mathcal{K}^q(T) := \left\{ u = u(t,x) : \|u\|_{\mathcal{K}^q(T')} < \infty \text{ for any } T' \in (0,T) \right\},$$

$$\|u\|_{\mathcal{K}^q(T)} := \sup_{t\in(0,T)} t^{\frac{d}{2}\left(\frac{1}{q_c} - \frac{1}{q}\right)} \|u(t)\|_{L^q}, \quad q > q_c := \frac{2d}{d-2}.$$

Local well-posedness (Chikami-Ikeda-T., arXiv:2009.07108.)

Given $u_0 \in \dot{H}^1(\mathbb{R}^d)$, there exists a time $T = T(u_0) > 0$ such that the Cauchy problem (HS) has a unique mild solution u on $[0, T]$. Moreover, denoting by $T_m = T_m(u_0)$ the maximal existence time of u, we have:

- $T_m < \infty \Longrightarrow \|u\|_{\mathcal{K}^q(T_m)} = \infty$;
- $T_m = \infty$ and $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0 \iff \|u\|_{\mathcal{K}^q(T_m)} < \infty$.

The proof is based on the standard fixed point argument and the following:

Linear estimates: Let $d \geq 1$, $\gamma \in [0, d)$ and $0 \leq \frac{1}{q_2} < \frac{\gamma}{d} + \frac{1}{p_1} < 1$. Then

$$\|e^{t\Delta}(|\cdot|^{-\gamma} f)\|_{L^{q_2}} \leq Ct^{-\frac{d}{2}(\frac{1}{q_1}-\frac{1}{q_2})-\frac{\gamma}{2}} \|f\|_{L^{q_1}}, \quad t > 0$$

(see Ben Slimene-Tayachi-Weissler (2017)).

Hardy-Sobolev inequality

$$\left(\int_{\mathbb{R}^d} |x|^{-\gamma} |f(x)|^{2^*(\gamma)} \, dx \right)^{\frac{1}{2^*(\gamma)}} \leq C_{\mathrm{HS}} \|f\|_{\dot{H}^1}, \quad f \in \dot{H}^1(\mathbb{R}^d),$$

where $C_{\mathrm{HS}} > 0$ is the best constant.

Ground state A non-trivial minimal energy solution of the stationary problem $-\Delta W = |x|^{-\gamma} W^{2^*(\gamma)-1}$ is called the *ground state*.

$$W(x) = \frac{c_{d,\gamma}}{(1 + |x|^{2-\gamma})^{\frac{d-2}{2-\gamma}}}$$

(see Aubin 1976, Talenti 1976, Lieb 1983).

Remark

- The cases $\gamma = 0, 2$ correspond to the Sobolev and Hardy inequalities, respectively.
- The equality holds in the Hardy-Sobolev inequality $\iff$ $f = \lambda^{\frac{d-2}{2}} W(\lambda x)$ ($\lambda > 0$).
- $E(W) = \frac{2-\gamma}{2(d-\gamma)} C_{\mathrm{HS}}^{-\frac{2(d-\gamma)}{2-\gamma}}$.

Theorem (Chikami-Ikeda-T., arXiv:2009.07108)

Let $d \geq 3$ and $\gamma \in (0, 2)$. Assume that $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $E(u_0) < E(W)$. Then, the solution u to (HS) satisfies the following:

(i) $\|u_0\|_{\dot{H}^1} \leq \|W\|_{\dot{H}^1} \implies u$ is dissipative.

(ii) $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1} \implies u$ blows up in finite time or at infinite time.
(If $u_0 \in L^2(\mathbb{R}^d)$ is additionally imposed, u must blow up in finite time).

Behavior of solutions

- u is dissipative $\overset{\text{def}}{\iff} T_m = \infty$ and $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0$
  $(\iff \|u\|_{\mathcal{K}^q(T_m)} < \infty)$.
- u blows up in finite time $\overset{\text{def}}{\iff} T_m < \infty$ $(\implies \|u\|_{\mathcal{K}^q(T_m)} = \infty)$.
- u blows up at infinite time $\overset{\text{def}}{\iff} T_m = \infty$ and $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = \infty$.

Known results (energy critical case)

- Kenig-Merle (2006, 2008): $\gamma = 0$, Schrödinger and wave equations.
- Ishiwata (2008): $\gamma = 0$ (p-Laplacian).
- Roxanas (2017): Harmonic map heat flow.
  (Gustafson-Roxanas (2018): $d = 4$ and $\gamma = 0$).
- Cho-Lee (2020): Similar study for Schrödinger equation.

$$i\partial_t u + \Delta u = -|x|^{-\gamma}|u|^{2^*(\gamma)-2}u, \quad t \in \mathbb{R},\ x \in \mathbb{R}^3$$

  ($d = 3$, $0 \le \gamma < \frac{3}{2}$, radial symmetry).

Our result

- No assumption on radial symmetry in our result. The problem becomes more difficult without this assumption.

Theorem

Assume that $E(u_0) < E(W)$. Then:

(i) $\|u_0\|_{\dot{H}^1} \le \|W\|_{\dot{H}^1} \Longrightarrow u$ is dissipative.

(ii) $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1} \Longrightarrow u$ blows up in finite time or at infinite time.

Strategy of proof of (i)

The proof is by contradiction and is based on concentration compactness & Rigidity by Kenig-Merle (2006).

Strategy of proof of (ii)

The proof is based on a method to reduce to an argument for an ordinary differential inequality (Levine's concavity method).

Here, let us give only an idea of proof of (i), and omit the proof of (ii).

Theorem (i)

$E(u_0) < E(W)$ and $\|u_0\|_{\dot{H}^1} \le \|W\|_{\dot{H}^1} \Longrightarrow \|u\|_{\mathcal{K}^q(T_m)} < \infty$.

Strategy of proof of (i)

The proof is by contradiction and is based on concentration compactness & Rigidity by Kenig-Merle (2006).

- Suppose (i) does not hold
  (i.e. $\exists u_0$ s.t. $E(u_0) < E(W)$, $\|u_0\|_{\dot{H}^1} \le \|W\|_{\dot{H}^1}$ and $\|u\|_{\mathcal{K}^q(T_m)} = \infty$).
- There exists a minimizing sequence $\{u_{0,n}\}_n$.
- We have a single profile ψ by the profile decomposition of $\{u_{0,n}\}_n$.
- The solution u^c to (HS) with $u^c(0) = \psi$ is a blow up solution with the minimal energy.
- The solution u^c must be 0 by the energy identity.
- Contradiction!

How to obtain a single profile ($J = 1$)

- (Profile decomposition) There exist $\{\psi^j\}_{j=1}^{\infty} \subset \dot{H}^1(\mathbb{R}^d)$, $\{\lambda_n^j\}_{j=1}^{\infty} \subset (0,\infty)$, $\{x_n^j\}_{j=1}^{\infty} \subset \mathbb{R}^d$ such that

$$u_{0,n}(x) = \sum_{j=1}^{J} \underbrace{\frac{1}{(\lambda_n^j)^{\frac{d-2}{2}}} \psi^j \left(\frac{x - x_n^j}{\lambda_n^j} \right)}_{=:\psi_n^j(x)} + w_n^J(x).$$

  Here, $J \geq 1$, $w_n^J \in \dot{H}^1(\mathbb{R}^d)$, $\lim_{J,n\to\infty} \|e^{t\Delta} w_n^J\|_{\mathcal{K}^q(\infty)} = 0$.

- $\{\lambda_n^j\}$ and $\{x_n^j\}$ satisfy one of the following:

  (a) $x_n^j \equiv 0$ or (b) $|x_n^j| \to \infty$ and $\frac{|x_n^j|}{\lambda_n^j} \to \infty$ as $n \to \infty$ for $j \geq 1$.

  - In the case (a), we can prove $J = 1$ by the same argument as $\gamma = 0$.
  - In the case (b), all solutions to (HS) with initial data ψ_n^j ($n \gg 1$) are dissipative by the following lemma; hence, the profiles that evolve into blow up solutions do NOT appear.

The following lemma plays an important role.

Lemma

Let $d \geq 3$ and $\gamma \in (0,2)$, and let u_n be a solution to (HS) with initial data

$$u_n(0) = \frac{1}{(\lambda_n)^{\frac{d-2}{2}}} \psi\left(\frac{x - x_n}{\lambda_n}\right), \quad \psi \in \dot{H}^1(\mathbb{R}^d)$$

Then, if $|x_n| \to \infty$ and $\frac{|x_n|}{\lambda_n} \to \infty$ $(n \to \infty)$, the solutions u_n are global and dissipative for $n \gg 1$.

Remark The above lemma does not hold in the case $\gamma = 0$.

- The problem (HS) with $\gamma = 0$ is invariant under the translation w.r.t x.
- $\|u_n\|_{\dot{H}^1} = \|u\|_{\dot{H}^1}$ for all $n \in \mathbb{N}$, where u is a solution with $u(0) = \psi$.

Idea of proof We expect the solutions u_n behave like the linear solutions if $n \gg 1$:

$$\partial_t u - \Delta u = |x - x_n|^{-\gamma} |u|^{2^*(\gamma)-2} u \sim 0 \quad \text{if } n \gg 1.$$

We first consider the absorbing case:

$$\begin{cases} \partial_t u - \Delta u = -|x|^{-\gamma}|u|^{2^*(\gamma)-2}u, & t > 0,\ x \in \mathbb{R}^d, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^d), \end{cases} \tag{HS-A}$$

where $d \geq 3$, $\gamma \in (0,2)$ and $2^*(\gamma) := \dfrac{2(d-\gamma)}{d-2}$.

Corollary 1 (Chikami-Ikeda-T., arXiv:2009.07108)

Let $d \geq 3$ and $\gamma \in (0,2)$. Then, the solution to (HS-A) with initial data in $\dot{H}^1(\mathbb{R}^d)$ is dissipative.

The proof is almost the same as in Theorem (i) with small modifications.

Let Ω is a domain and $0 \in \Omega$. We next consider the Dirichlet problem:

$$\begin{cases} \partial_t u - \Delta u = |x|^{-\gamma}|u|^{2^*(\gamma)-2}u, & t>0,\ x \in \Omega, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0 \in H_0^1(\Omega)\ (= \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}), \end{cases} \tag{HS-D}$$

where $d \geq 3$, $\gamma \in (0,2)$ and $2^*(\gamma) := \dfrac{2(d-\gamma)}{d-2}$.

Corollary 2 (Chikami-Ikeda-T., arXiv:2009.07108)

Let $d \geq 3$ and $\gamma \in (0,2)$. Assume that $u_0 \in \dot{H}_0^1(\Omega)$ with $E_\Omega(u_0) < E(W)$.
Then, the solution u to (HS-D) satisfies the following:

(i) $\|u_0\|_{\dot{H}^1(\Omega)} \leq \|W\|_{\dot{H}^1} \implies u$ is dissipative.

(ii) $\|u_0\|_{\dot{H}^1(\Omega)} > \|W\|_{\dot{H}^1} \implies u$ blows up in finite time.

Remark $\|\cdot\|_{\dot{H}^1(\Omega)}$ and $E_\Omega(\cdot)$ replace the integral range from $\mathbb{R}^d$ to Ω.

Summary

1. (Hardy-Sobolev parabolic equation)

$$\begin{cases} \partial_t u - \Delta u = |x|^{-\gamma}|u|^{2^*(\gamma)-2}u, & t > 0,\ x \in \mathbb{R}^d, \\ u(0) = u_0 \in \dot{H}^1(\mathbb{R}^d). \end{cases} \tag{HS}$$

2. (HS) is LWP in $\dot{H}^1(\mathbb{R}^d)$.

3. (Theorem) $E(u_0) < E(W)$.
   (i) $\|u_0\|_{\dot{H}^1} \leq \|W\|_{\dot{H}^1} \Rightarrow u$ is dissipative.
   (ii) $\|u_0\|_{\dot{H}^1} > \|W\|_{\dot{H}^1} \Rightarrow u$ blows up in finite time or at infinite time.

4. (Strategy of proof of Theorem)
   (i) Concentration compactness & Rigidity
   (How to deal with $\{x_n^j\}$ in the profile decomposition).
   (ii) Levine's concavity method.

5. The absorbing case & the Dirichlet problem.

Thank you for your attention