Global well-posedness for the non-linear Maxwell-Schrödinger system

Raffaele Scandone (GSSI L'Aquila)

13th ISAAC congress, *Harmonic Analysis and PDEs* Session

4 August 2021

Based on joint work with P. Antonelli and P. Marcati

Maxwell-Schrödinger system

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\,\Delta_A\psi + \phi\psi \\ -\Delta\phi - \partial_t \operatorname{div} A = \rho \\ \Box A + \nabla(\partial_t\phi + \operatorname{div} A) = J \end{cases} \qquad (MS)$$

in the unknown $(\psi, \phi, A) : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{C} \times \mathbb{R} \times \mathbb{R}^3$, where

- $\Delta_A := \nabla_A^2 = (\nabla - iA)^2$ is the *magnetic Laplacian*.
- (ϕ, A) is the electromagnetic potential.
- $\rho := |\psi|^2$, $J := \operatorname{Im}(\overline{\psi}\nabla_A\psi)$ are the *charge* and *current* densities.

Conserved quantities: *charge* $\mathcal{Q} := \|\psi\|_{L^2}^2$, and *energy*

$$\mathcal{E} := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_A\psi|^2 + |\nabla\phi|^2 + |\partial_t A|^2 + |\operatorname{rot} A|^2 dx.$$

Gauge invariance: $(\psi, \phi, A) \mapsto (e^{i\lambda}\psi, \phi - \partial_t\lambda, A + \nabla\lambda), \quad \lambda : \mathbb{R}^3 \to \mathbb{R}.$

In the *Coulomb gauge*, i.e. div $A = 0$, (MS) takes the form

$$\text{(MS)} \quad \begin{cases} i\partial_t \psi = -\frac{1}{2}\Delta_A \psi + \phi\psi \\ \Box A = \mathbb{P}J, \end{cases}$$

where $\phi := (-\Delta)^{-1}|\psi|^2$ (Hartree potential), and $\mathbb{P} := \mathbb{I} - \nabla \operatorname{div}(-\Delta)^{-1}$ is the *Helmholtz-Leray projection* ($\operatorname{div}\mathbb{P} = 0$).

$$\mathcal{E} := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_A \psi|^2 + |\partial_t A|^2 + |\nabla A|^2 + |\nabla \phi|^2 dx.$$

Using (ρ, J) as unknown (Madelung transform), we formally get (issues: justify the derivation, possible presence of vacuum regions):

$$\text{(QMHD)} \quad \begin{cases} \partial_t \rho + div\ J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) = \rho E + J \wedge B + \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases}$$

(E, B) satisfies Maxwell equations, $\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right)$ is the Bohm potential.

Nonlinear Maxwell-Schrödinger system (Coulomb gauge)

$$(\gamma\text{-MS}) \quad \begin{cases} i\partial_t\psi = -\frac{1}{2}\,\Delta_A\psi + \phi\psi + |\psi|^{\gamma-1}\psi \\ \Box A = \mathbb{P}J \end{cases}$$

Conserved quantities: *charge* $\mathcal{Q} := \|\psi\|_{L^2}^2$, and *energy*

$$\mathcal{E}(t) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_A\psi|^2 + |\partial_t A|^2 + |\nabla A|^2 + |\nabla\phi|^2 + \frac{2}{\gamma+1}|\psi|^{\gamma+1}dx$$

Using Madelung transform, formally we get the (γ-QMHD) system

$$\begin{cases} \partial_t\rho + div\ J = 0 \\ \partial_t J + \mathrm{div}\left(\frac{J\otimes J}{\rho}\right) + \nabla P(\rho) = \rho E + J\wedge B + \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases}$$

$P(\rho) = \frac{\gamma-1}{\gamma+1}\rho^{\frac{\gamma+1}{2}}$ is an isentropic *pressure* term.

Magnetic Sobolev norms: $\|f\|_{H^s_A} := \|\mathcal{D}_A^s f\|_{L^2}$, $\mathcal{D}_A := (1-\Delta_A)^{1/2}$.

Assume $A \in \dot{H}^1$. Then $H^s_A \approx H^s$ for $s \in [-2, 2]$.

Energy space: $(\psi, A, \partial_t A) \in M^{1,1} := H^1 \times H^1 \times L^2$.

Consider more generally $M^{s,\sigma} := H^s \times H^\sigma \times H^{\sigma-1}$, $s \in [1,2]$, $\sigma \geq 1$.

Goals

- Local and global well-posedness in $M^{s,\sigma}$ for (γ-MS);
- Existence and stability of solutions to (γ-QMHD).

Main obstacles

- Presence of a *time-dependent* magnetic Laplacian:
  -even the well-posedness of the *linear* problem is hard!
  -lack of Strichartz estimates (even for time *independent* potential, one needs smallness or spectral assumptions at zero energy).
  Hard to control the pure-power term.
- Derivative term in the current density J.

A priori estimates - Part 1

- Koch-Tzvetkov type estimates for Schrödinger equation

  Let ψ be a solution to $i\partial_t\psi = -\Delta\psi + F$. Then

$$\|\psi\|_{L^2_T B^{s-\alpha}_{6,2}} \lesssim \|\psi\|_{L^\infty_T H^s} + T^{1/2}\|F\|_{L^2_T H^{s-2\alpha}}, \quad \alpha > 0.$$

  -IDEA: localization of ψ, F in $[0,T] \times \widehat{\mathbb{R}^3}$ + Standard Strichartz.
  -When $\alpha \geq 1/2$, it allows to handle the *derivative* term $A \cdot \nabla$ appearing in the expansion of Δ_A.

- Koch-Tzvetkov type estimates for magnetic Schrödinger.

  Suppose that $A \in L^q_T W^{1-2/q,r}$, $\forall\,(q,r)$ s.t. $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $q \in [2,\infty]$. Let $s \in [1,2]$, and ψ a solution to $i\partial_t\psi = -\Delta_A\psi + F$. Then

$$\|\psi\|_{L^2_T B^{s-\alpha}_{6,2}} \lesssim_T C_A\Big(\|\psi\|_{L^\infty_T H^s} + \|F\|_{L^2_T H^{s-2\alpha}}\Big), \quad \alpha \geq 1/2.$$

A priori estimates - Part 2

- Strichartz estimates for $(\Box + 1)A = F$.

$$\max_{k=0,1} \|\partial_t^k A\|_{L^q_T W^{s-k-2/q,r}} \lesssim \|(A_0, A_1)\|_{H^s \times H^{s-1}} + \|F\|_{L^{q_0'}_T W^{s+2/q_0-1,r_0'}}.$$

Admissible pair: $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, $q \in (2, \infty]$. The endpoint case $(q, r) = (2, \infty)$ fails in general. It holds for *radial* data, or as soon as we have some extra *angular regularity*.

- Product estimates. Let $s \geq 0$, and $\frac{1}{p_i} + \frac{1}{q_i} = \frac{1}{p}$, $i = 1, 2$. Then

$$\|\mathbb{P}(\overline{\psi}_1 \nabla \psi_2)\|_{W^{s,p}} \lesssim \|\psi_1\|_{W^{s,p_1}} \|\nabla \psi_2\|_{L^{q_1}} + \|\nabla \psi_1\|_{L^{q_2}} \|\psi_2\|_{W^{s,p_2}}.$$

-IDEA: exploits $\mathbb{P}\nabla = 0$. Allows to handle the derivative term in $\mathbb{P}J$.
-This idea is **not enough** when there are *pure-curl* currents, like the spin-current in the Maxwell-Pauli system.

A priori estimates for (γ-MS)

Given $\sigma > 1$ Strichartz for Klein Gordon + product estimate gives

$$\|A\|_{L^2_T L^\infty} \lesssim_T \langle \|(\psi, A, \partial_t A)\|^m_{L^\infty_T M^{1,1}} \rangle \langle \|(A_0, A_1)\|^m_{H^\sigma \times H^{\sigma-1}} \rangle.$$

(It's enough $\sigma = 1$ with "endpoint initial data"). For $\gamma \in (1,4)$

$$\||\psi|^{\gamma-1}\psi\|_{L^2_T H^{s-1}} \lesssim \|\psi\|^{\gamma-1}_{L^\infty_T H^1} \|\psi\|_{L^2_T W^{1/2-\varepsilon(\gamma),6} \cap L^\infty_T H^1}$$

Hence, by Koch-Tzvetkov estimates, previous bound, and *bootstrap*, we obtain that for $s \in [1,2]$, $\sigma > 1$

$$\|\psi\|_{L^2_T B^{s-1/2}_{6,2}} \lesssim_T \langle \|(\psi, A, \partial_t A)\|^n_{L^\infty_T M^{1,1}} \rangle \langle \|(A_0, A_1)\|^n_{H^\sigma \times H^{\sigma-1}} \rangle \|\psi\|_{L^\infty_T H^s}.$$

Consequences: Let (ψ, A) a *finite energy*, weak solution to (γ-MS) with $(u_0, A_0, A_1) \in M^{1,\sigma}$, $\sigma > 1$. Then

- $\|\psi\|_{L^2_T B^{1/2}_{6,2}} + \|A\|_{L^2_T L^\infty} \lesssim_T 1.$
- The Lorentz force $F := \rho E + J \wedge B$ is well-defined and belongs to $L^2_T L^1$ ($M^{1,1}$ alone is not sufficient).

Local well-posedness for (γ-MS)

- Let $s \in [\frac{11}{8}, 2]$, $\sigma \in (1,3)$, $\gamma \in (s, \gamma^*)$, for $\gamma^* := \gamma^*(s) \in (3, \infty]$ (when $s = 2$, $\gamma \in (1, \infty)$). Then (γ-MS) il LWP in $M^{s,\sigma}$.
- Holds also when $\sigma = 1$ with "endpoint initial data".
- Relatively easy at high regularity ($s > \frac{3}{2}$).
- Exploits a priori estimates at intermediate regularity ($s \in (\frac{11}{8}, 2)$).
- Open problem at low regularity, in particular in the energy space $M^{1,1}$ (existence of **weak** solutions in $M^{1,1}$, for $\gamma \in (1,5)$, can be proved through a regularization/compactness argument).
- For the **linear** Maxwell-Schrödinger system, well-posedness in the energy space has been proved in [Bejenaru-Tataru, 2009]. Extending their analysis to the non-linear case is non-trivial.

Global well-posedness (Part I)

(Generalized) Brezis-Gallouet inequality:

$$\|f\|_{L^\infty} \lesssim 1 + \|f\|_{B^{3/p}_{p,r}} \ln_+^{\frac{r-1}{r}} \|f\|_{W^{3/q+\varepsilon,q}}$$

A standard energy methods yields

$$\|\psi\|_{L^\infty_T H^s} \lesssim_T 1 + \int_0^T \|\psi(t)\|_{L^\infty}^{\gamma-1} \|\psi\|_{L^\infty_t H^s}$$

Assume $s > \frac{3}{2}$, $\gamma \in (s, 3)$. Brezis-Gallouet for $B^{1/2}_{6,2}$ yields

$$\|\psi\|_{L^\infty_T H^s} \lesssim_T 1 + \int_0^T \|\psi(t)\|_{B^{1/2}_{6,2}}^{\gamma-1} \|\psi\|_{L^\infty_t H^s} \ln_+^{\frac{\gamma-1}{2}} \|\psi\|_{L^\infty_t H^s}.$$

Uniform bound on $L^2_T B^{1/2,6}$ + Grönwall imply GWP with exp exp-bounds.

- The method can be adapted to cover $s \in (1, \frac{3}{2}]$, as $B^{s-1/2}_{6,2} \hookrightarrow L^\infty$.

Modified higher order energy

Let ψ be a solution to (γ-MS), $\gamma > 2$. Define

$$\mathcal{E}_2(t) := \int_{\mathbb{R}^3} |\partial_t \psi|^2 - (\gamma - 1)|\psi|^{\gamma-1}|\nabla|\psi||^2 - \frac{\gamma - 1}{\gamma}|\psi|^{2\gamma} dx$$

Same functional used by [Planchon-Tzvetkov-Visciglia, 2016] to prove polynomial growth of H^2-norm for sub-cubic NLS on compact manifolds.

- $\mathcal{E}_2(t)$ is *equivalent* to $\|\psi(t, \cdot)\|^2_{H^2}$

$$\left| \, \mathcal{E}_2(t) - \|\psi\|^2_{H^2_A} \, \right| \lesssim \langle t \rangle^n \langle \|\psi\|_{H^2} \rangle^{c(\gamma)}, \quad c(\gamma) < 2.$$

- Gain of regularity when computing the derivative of $\mathcal{E}_2$:

$$\frac{d}{dt}\mathcal{E}_2(t) = c(\gamma) \int_{\mathbb{R}^3} |\psi|^{\gamma-2}\partial_t|\psi||\nabla_A \psi|^2 dx + \text{ "lower order terms"}$$

Global well-posedness (Part II)

Using the a-priori estimates we find $\varepsilon(\gamma) > 0$ such that

$$\int_0^T \frac{d}{dt}\mathcal{E}_2(t)\,dt \lesssim \|\psi\|_{L^2_T B^{1/2}_{6,2}} \|\psi\|^{2-\varepsilon(\gamma)}_{L^\infty_T H^2}, \quad T \in (0,1)$$

Using $\mathcal{E}_2(t) \approx \|u(t,\cdot)\|^2_{H^2}$ and uniform bound on $L^2_T B^{1/2,6}$ we get

$$\|u(T,\cdot)\|^2_{H^2} - \|u(0,\cdot)\|^2_{H^2} \lesssim \|u\|^{2-\varepsilon(\gamma)}_{L^\infty_T H^2}, \quad T \in (0,1)$$

which yields

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{H^2} \lesssim T^{\frac{1}{\varepsilon(\gamma)}}, \quad T > 0.$$

- GWP in $M^{2,\sigma}$, for $\gamma \in (2,3)$, with **polynomial** bounds.
- When $\gamma = 3$, one can get GWP (with exponential bounds) by means of an iteration argument.

Refined Strichartz estimates

Let χ_α a bump function, with $\chi_\alpha \equiv 1$ on the unit cube of center $\alpha \in \mathbb{Z}^3$.

- Refined Strichartz for $(\Box + 1)A = F$.

$$\max_{k=0,1} \|\chi_\alpha \partial_t^k A\|_{\ell^2_\alpha L^q_T W^{s-k-2/q,r}} \lesssim \|(A_0, A_1)\|_{H^s \times H^{s-1}} + \|F\|_{L^{q_0'}_T W^{s+2/q_0-1,r_0'}}.$$

Exploits the finite speed of propagation for wave equation.

- Refined Koch-Tzvetkov for $i\partial_t \psi = -\Delta_A \psi + F$.

Suppose that $\chi_\alpha A \in \ell^2_\alpha L^q_T W^{1-2/q,r}$, for all admissible pairs (q, r). Let $s \in [0, 2)$, $\theta \in (0, 1)$, and $m \in \left(\frac{\theta-1}{2}, \frac{2\theta-1}{2}\right)$. Then

$$\|\chi_\alpha \mathcal{D}_A^{s-\theta} \psi\|_{\ell^2_\alpha L^2_T W^{m,6}} \lesssim_T C_A \Big(\|\psi\|_{L^\infty_T H^s} + \|F\|_{L^2_T H^{s-2\theta+2m}} \Big).$$

Gain in term of summability w.r.t. spatial localization.

Local smoothing estimates

- Fix $\sigma \in (1, \frac{7}{6})$, $\gamma \in (1,4)$. Let (ψ, A) be a solution to (γ-MS), with initial data $(\psi_0, A_0, A_1) \in M^{1,\sigma}$. Then, for every $\delta \in (1, \sigma - 1)$, we have the local-smoothing estimate

$$\|\chi_\alpha \psi\|_{\ell^\infty_\alpha L^2_T H^{1+\delta}} \lesssim_T \langle \|(\psi, A)\|^n_{L^\infty_T (H^1 \times H^1)} \rangle \langle \|(A_0, A_1)\|^n_{H^\sigma \times H^{\sigma-1}} \rangle.$$

- IDEA OF THE PROOF:
  - Let h increas., bdd, such $h'(t) = 1$ for $|t| \geq \frac{1}{2}$, $h'(t) = 0$ for $|t| \geq 1$.
  -Given $\alpha \in \mathbb{Z}^3$ and spatial direction j, set $h_{\alpha,j}(x) := h(x_j - \alpha_j)$
  -define the smoothing operator $L_{\alpha,j} := \mathcal{D}_A^{\delta-1} h_{\alpha,j} (\partial_j - iA_j) \mathcal{D}_A^{\delta-1}$.
  -Use energy method for the equation satisfied by $L_{\alpha,j} \mathcal{D}_A \psi$, together with commutator bounds and the refined Koch-Tzvetkov estimate.
- Local smoothing allows to improve a bit the range of (s, σ) in the LWP for (γ-MS).

Existence and stability of solutions to (γ-QMHD)

- We are interested in *weak* solutions, obtained as Madelung transform ($\rho = |\psi|^2$, $J = Im(\overline{\psi}\nabla_A\psi)$) from $M^{1,\sigma}$-solutions to (γ-MS).
- "Natural" hydrodyn. variables: $\sqrt{\rho}$ and $\Lambda := Im(\overline{\varphi}\nabla_A\psi)$, where $\psi = \varphi|\psi|$. The map $\psi \mapsto (\sqrt{\rho}, \Lambda)$ is continuous from H^1 to $H^1 \times L^2$.
- We do **not** have GWP in $M^{1,\sigma}$ for (γ-MS). Hence we cannot use a standard approximation/stability approach.
- We start from a *weak* $M^{1,\sigma}$ solution to (γ-MS), $\sigma > 1$, and we directly manipulate the integral (Duhamel) formulations. A priori estimate allows to *justify* all the passages, when $\gamma \in (1,4)$.
- Weak stability of $(\sqrt{\rho}, \Lambda)$ obtained using the local smoothing.
- QUESTION: Can we cover $\gamma \in [4,5)$? Can we cover the case $\sigma = 1$? (Same problem arises also in classical MHD model).

Thank you for your attention!