Direct and inverse Cauchy problems for space–time fractional differential equations

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Outline

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Fractional calculus

What is fractional calculus? The main idea is to generalize the order of differentiation and integration outside just the set of whole numbers.

For an n-fold integral there is a well known formula

$$\int_a^x \int_a^{x_1} \dots \int_a^{x_{n-2}} \int_a^{x_{n-1}} \varphi(x_n) dx_n dx_{n-1} \dots dx_1 = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} \varphi(t) dt.$$

Hence it was formulated the RL fractional integral

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad 0 < \alpha < 1,$$

and the inverse operator, i.e. RL fractional derivative as

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} f(t) dt, \quad 0 < \alpha < 1,$$

while the Caputo fractional derivative is ${}^{C}D_{a+}^{\alpha} f(x) = I_{a+}^{1-\alpha} f'(x)$.

General definitions

Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $-\infty \leqslant a < b \leqslant +\infty$. Let g be an integrable function on $[a, b]$, and let $\varphi \in C^1[a, b]$ be such that $\varphi'(t) > 0$ for all $t \in [a, b]$. The left-sided RL fractional integral of a function g with respect to another function φ is defined by

$$\mathbb{I}_{a+}^{\alpha,\varphi} g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \varphi'(t)(\varphi(x) - \varphi(t))^{\alpha-1} g(t) dt.$$

While, the left-sided RL fractional derivative of a function g with respect to another function φ is defined by

$$\mathbb{D}_{a+}^{\alpha,\varphi} g(x) = \left(\frac{1}{\varphi'(x)} \frac{d}{dx} \right)^n (\mathbb{I}_{a+}^{n-\alpha,\varphi} g)(x),$$

where $\varphi \in C^n[a, b]$ with $\varphi'(t) > 0$ for all $t \in [a, b]$ and $n = \lfloor \mathrm{Re}(\alpha) \rfloor + 1$.

Some classical books on the field

Some sources:

S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives: Theory and Ap- plications, Gordon Breach Science Publishers, Yverdon, 1993. Translated from Russian: Nauka i Tekhnika, Minsk, 1987.

A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

Another introductory textbook:
K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.

Starting point

Several researchers have investigated fundamentals solutions of the multidimensional fractional diffusion–wave equation ($a = 0$)

$$\begin{aligned} ({}^C D_t^{\beta} + a\, {}^C D_t^{\alpha} - c^2 \Delta_x) h(x,t) &= 0, \\ h(x,0) &= \delta(x), \\ \frac{h(x,0)}{\partial t} &= \delta(x), \end{aligned}$$

and time-fractional telegraph equation ($a > 0$), where $x \in \mathbb{R}^n$, $t > 0$, $0 < \alpha \leqslant 1$, $1 < \beta \leqslant 2$, $a \geqslant 0$, $c > 0$, $\Delta_x = \sum_{k=1}^n \partial_{x_k}^2$, $\delta(x) = \prod_{i=1}^n \delta(x_i)$ is the distributional Dirac delta function in $\mathbb{R}^n$.

Starting point

Notice that:

1. ${}^{C}D_t^{\beta} - c^2\Delta_x$ becomes $(\beta = 0)$ to the Helmholtz operator $I - c^2\Delta_x$.
2. ${}^{C}D_t^{\beta} - c^2\Delta_x$ becomes $(\beta = 1)$ to the Heat operator $\partial_t - c^2\Delta_x$.
3. ${}^{C}D_t^{\beta} - c^2\Delta_x$ becomes $(\beta = 2)$ to the wave operator $\partial_{tt}^2 - c^2\Delta_x$.

For some related works, see e.g.:

1. W.R.Schneider, W. Wyss. Fractional diffusion and wave equations. J. Math. Phys., 30(1), (1989), 134–144.
2. A. Hanyga, Multidimensional solutions of time-fractional diffusion-wave equations, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci., 458, (2002), 933–957.

Recent related works

Recent results gave explicit solutions in a close form by using some special functions:

1. Y.Luchko. Multi-dimensional fractional wave equation and some properties of its fundamental solution. Commun. Appl. Ind. Math., 6(1), (2014) # 485, 21p.
2. M. Ferreira, N. Vieira. Fundamental solutions of the time fractional diffusion–wave and parabolic Dirac operators. J. Math. Anal. Appl., 447(1), (2017), 329–353.

New applications in Clifford Analysis

1. Ferreira M, Rodrigues MM, Vieira N. Fundamental solution of the time–fractional telegraph Dirac operator. Math. Meth. Appl. Sci. 40(18), (2017), 7033–7050.
2. Ferreira M, Rodrigues MM, Vieira N. First and Second Fundamental Solutions of the Time–Fractional Telegraph Equation with Laplace or Dirac Operators. Adv. Appl. Clifford Algebras. 28(42), (2018).

Base questions: Published works and future works

1. $({}^{C}D_t^{\beta} - c^2\Delta_x)h(x,t) = 0$ Done for RL and Caputo! Open for some other fractional differential operators like Prabhakar !
2. $({}^{C}D_t^{\beta} + a\,{}^{C}D_t^{\alpha} - c^2\Delta_x)h(x,t) = 0$ Done for RL and Caputo ! Open for some other fractional differential operators like Prabhakar !

For (1) and (2), a classical approach is applied the space-Fourier transform in $\mathbb{R}^n$ to get a fractional initial-value problem of constant coefficients in the time-variable, whose solution is known [1]. And then, taking the inverse Fourier transform. The "most difficult" and "interesting" part is obtained a close form of the solutions.

[1] Y. Luchko, R. Gorenflo. An operational method for solving fractional differential equations with the Caputo derivatives. Acta Math Vietnam. 24(2), (1999), 207–233.

Base questions: Published works and future works

1. $\left({}^{C}D_t^{\beta_0} + a_1\,{}^{C}D_t^{\beta_1} + \cdots + a_m\,{}^{C}D_t^{\beta_m} - c^2\Delta_x\right)h(x,t) = 0$
   Partially known! The main problem is to obtain a close solution.
   Similar approach ! New objects to be analyzed ! Use different fractional operators, like Prabhakar !
2. $\left({}^{C}D_t^{\beta_0} + a_1(t)\,{}^{C}D_t^{\beta_1} + \cdots + a_m(t)\,{}^{C}D_t^{\beta_m} - c^2\Delta_x\right)h(x,t) = 0$
   Under study! Let us now discuss it...

A classical way for the solution

We give the main steps for getting a "first approach" of a "suitable solution"[2] of the following space-time fractional Cauchy problem:

$$\left({}^{C}D_t^{\beta_0} + a_1(t)\,{}^{C}D_t^{\beta_1} + \cdots + a_m(t)\,{}^{C}D_t^{\beta_m} - c^2\Delta_x\right)h(x,t) = 0.$$

By applying the space-Fourier transform we get:

$$\left({}^{C}D_t^{\beta_0} + a_1(t)\,{}^{C}D_t^{\beta_1} + \cdots + a_m(t)\,{}^{C}D_t^{\beta_m} + c^2|s|^2\right)\mathcal{F}(h)(s,t) = 0.$$

[2] well-posed: Uniquely determined solution that depends continuously on its data

So, we have to solve an ordinary FDE with variable coefficients ! For instance, for fractional derivatives with respect to another function, explicit solutions were published in:

1. J.E. Restrepo, M. Ruzhansky, D. Suragan, "Explicit solutions for linear variable-coefficient fractional differential equations with respect to functions", Appl. Math. Comput., 403, # 126177, (2021).
2. J.E. Restrepo, D. Suragan, "Hilfer-type (with respect to functions) fractional differential equations with variable coefficients", Chaos Solitons Fractals, 143, # 111146, (2021).

Therefore, we get a solution by applying the inverse space-Fourier transform. Still some open questions here with respect to a close form of the solution !

Form of the solutions for FDEs with variable coefficients

$$ {}^{C}D_{0+}^{\beta_0,\phi}x(t) + \sum_{i=1}^{m} \Theta_i(t)\,{}^{C}D_{0+}^{\beta_i,\phi}x(t) = h(t), \quad t \in [0,T],\ m \in \mathbb{N}, \tag{1} $$

under the initial conditions

$$ \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^k x(t)\Big|_{t=+0} = 0, \quad k = 0,1,\ldots,n_0-1, \tag{2} $$

where $\beta_i \in \mathbb{C}$, $\mathrm{Re}(\beta_i) > 0$, $\mathrm{Re}(\beta_0) > \mathrm{Re}(\beta_1) > \ldots > \mathrm{Re}(\beta_m) \geqslant 0$ and $n_i - 1 \leqslant \mathrm{Re}(\beta_i) < n_i$, $n_i = \lfloor \mathrm{Re}\beta_i \rfloor + 1$, $i = 0,1,\ldots,m$.

Theorem

Let $h, \Theta_i \in C[0,T]$, $i = 1,\ldots,m$. Then the initial value problem (1) and (2) has a unique solution $x \in C^{n_0-1,\beta_0}[0,T]$ and it is given by the formula

$$ x(t) = \sum_{k=0}^{+\infty} (-1)^k I_{0+}^{\beta_0,\phi}\left(\sum_{i=1}^{m} \Theta_i(t) I_{0+}^{\beta_0-\beta_i,\phi}\right)^k h(t). $$

Space-time fractional Cauchy problem

We consider the following space-time fractional Cauchy problem (FCP):

$$
\begin{aligned}
{}^{C}\partial_t^{\beta_0,\phi} w(x,t) + \sum_{i=1}^{m-1} \Theta_i(t)\, {}^{C}\partial_t^{\beta_i,\phi} w(x,t) + \Theta_m(t)(-\Delta)^{\lambda} w(x,t) & \\
= h(x,t),\ \ t \in (0,T],\ \ x \in \mathbb{R}^n&, \\
w(x,t)\big|_{t=0} = w_0(x)&, \\
\partial_t w(x,t)\big|_{t=0} = w_1(x)&, \\
\vdots & \\
\partial_t^{n_0} w(x,t)\big|_{t=0} = w_{n_0-1}(x)&,
\end{aligned}
$$

where $0 < \lambda \leqslant 1$, $\beta_i \in \mathbb{C}$, $\Theta_i(t) \in C[0,T]$, $n_i = \lfloor \mathrm{Re}\beta_i \rfloor + 1$, $i = 1, \ldots, m$, and $\mathrm{Re}(\beta_0) > \mathrm{Re}(\beta_1) > \ldots > \mathrm{Re}(\beta_{m-1}) > 0$. Here $h(t,\cdot) \in C[0,T]$.

Solution of the FCP

Theorem (FCP)

The (FCP) has a unique solution given by [a]:

$$w(x,t) = \sum_{j=0}^{n_0-1} w_j(x)\frac{t^j}{j!} + \sum_{j=0}^{n_0-1} \int_{\mathbb{R}^n} \left(\mathcal{F}_s^{-1} H_j(t, |s|^2, \Theta_1, \dots, \Theta_m)\right)(x-y) w_j(y) dy.$$

[a] J.E. Restrepo, D. Suragan, "Direct and inverse Cauchy problems for generalized space-time fractional differential equations", Adv. Differential Equations, 26(7–8), (2021), 305–339.

In the case of one variable coefficient, a different expression for the solution is established in: A. Fernandez, J.E. Restrepo, D. Suragan, "A new representation for the solutions of fractional differential equations with variable coefficients", arXiv:2105.00870v1, 2021.

The classical heat equation

We begin with the initial value problem for the heat equation on $\mathbb{R}^n$:

$$\begin{cases} w_t(x,t) = \Delta_x w(x,t), \quad x \in \mathbb{R}^n, \ t > 0, \\ w(x,t)\big|_{t=0+} = g(x), \end{cases}$$

where $g, \widehat{g} \in L^1(\mathbb{R}^n)$. The classical solution of this equation can be found by applying the space Fourier transform and then solving the transform equation, which is an ordinary differential equation with respect to the variable t. So, the solution is given by:

$$w(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \widehat{g}(y) e^{-|y|^2 t} e^{-ix\cdot y} dy.$$

The classical heat equation (using our approach)

On the other hand, by Theorem FCP with $\beta_0 = 1$, $\Theta_i(t) \equiv 0$ for $i = 1, \dots, m-1$, $\Theta_m(t) = 1$, $h(x,t) \equiv 0$ and $w(x,t)\big|_{t=0+} = g(x)$, we have that the solution of the heat equation is given by:

$$w(x,t) = g(x) + \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} K_0(t, |s|^2, 1)(y)\widehat{g}(y)dy,$$

where

$$\begin{aligned} K_0(t, |s|^2, 1) &= \sum_{k=0}^{+\infty} (-1)^{k+1} \left(I_{0+}^1 \left(|s|^2 I_{0+}^1 \right)^k \right) (|s|^2 t^0) \\ &= \sum_{k=1}^{+\infty} (-|s|^2)^k \frac{t^k}{k!} = -1 + \sum_{k=0}^{+\infty} \frac{(-|s|^2 t)^k}{k!} = -1 + e^{-|s|^2 t}. \end{aligned}$$

The classical heat equation (using our approach)

This implies

$$w(x,t) = g(x) + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}(-1 + e^{-|y|^2 t})\widehat{g}(y)dy$$
$$= g(x) - g(x) + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y} e^{-|y|^2 t}\widehat{g}(y)dy,$$

which coincides with the classical one.

The classical wave equation

We now focus on the initial value problem for the wave equation on $\mathbb{R}^n$:

$$\begin{cases} \partial_{tt}^2 w(x,t) = \Delta_x w(x,t), \quad x \in \mathbb{R}^n, \ t > 0, \\ w(x,t)\big|_{t=0+} = f(x), \\ \partial_t w(x,t)\big|_{t=0+} = g(x), \end{cases}$$

where $f, g, \widehat{f}, \widehat{g} \in L^1(\mathbb{R}^n)$ such that $\widehat{f}(y)\cos(|y|t)$, $\widehat{g}(y)\frac{\sin(|y|t)}{|y|} \in L^1(\mathbb{R}^n)$ for any $y \in \mathbb{R}^n$ and $t \geqslant 0$. The solution of wave equation can be found by applying the space Fourier transform, and hence solving an ordinary differential equation in the variable t. Indeed, the solution is represented by:

$$w(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ix\cdot y} \left(\widehat{f}(y)\cos(|y|t) + \widehat{g}(y)\frac{\sin(|y|t)}{|y|} \right) dy.$$

The classical heat equation (using our approach)

By using Theorem FCP with $\beta_0 = 2$, $\Theta_i(t) \equiv 0$ for $i = 1, \dots, m-1$, $\lambda = 1$, $\Theta_m(t) = 1$, $h(x,t) \equiv 0$, $w(x,t)\big|_{t=0+} = f(x)$ and $\partial_t w(x,t)\big|_{t=0+} = g(x)$, we obtain

$$\begin{aligned} w(x,t) &= f(x) + g(x)t + \int_{\mathbb{R}^n} \mathcal{F}^{-1}\big(K_0(t,|s|^2,1)\big)(x-y)f(y)dy \\ &\quad + \int_{\mathbb{R}^n} \mathcal{F}^{-1}\big(K_1(t,|s|^2,1)\big)(x-y)g(y)dy \\ &= f(x) + g(x)t + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y} K_0(t,|s|^2,1)(y)\widehat{f}(y)dy \\ &\quad + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y} K_1(t,|s|^2,1)(y)\widehat{g}(y)dy, \end{aligned}$$

The classical wave equation (using our approach)

where

$$K_0(t,|s|^2,1)=\sum_{k=0}^{+\infty}(-1)^{k+1}\left(I_{0+}^2\left(|s|^2 I_{0+}^2\right)^k\right)(|s|^2 t^0)$$

$$=\sum_{k=0}^{+\infty}(-|s|^2)^{k+1}\frac{t^{2(k+1)}}{(2(k+1))!}=-1+\sum_{k=0}^{+\infty}\frac{(-|s|t)^{2k}}{(2k)!}=-1+\cos(|s|t),$$

and

$$K_1(t,|s|^2,1)=\sum_{k=0}^{+\infty}(-1)^{k+1}\left(I_{0+}^2\left(|s|^2 I_{0+}^2\right)^k\right)(|s|^2 t^1)$$

$$=\sum_{k=0}^{+\infty}(-|s|^2)^{k+1}\frac{t^{2(k+1)+1}}{(2(k+1)+1)!}$$

$$=-t+\frac{1}{|s|}\sum_{k=0}^{+\infty}\frac{(-1)^k(|s|t)^{2k+1}}{(2k+1)!}=-t+\frac{\sin(|s|t)}{|s|}.$$

The classical wave equation (using our approach)

It yields

$$
\begin{aligned}
w(x,t) = f(x) + g(x)t &+ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}(-1+\cos(|y|t))\widehat{f}(y)dy \\
&+ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\left(-t+\frac{\sin(|y|t)}{|y|}\right)\widehat{g}(y)dy, \\
= f(x) + g(x)t - f(x) &+ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\cos(|y|t)\widehat{f}(y)dy \\
&- tg(x) + \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-ix\cdot y}\frac{\sin(|y|t)}{|y|}\widehat{g}(y)dy
\end{aligned}
$$

which coincides with the classical one.

Example 1

Let us consider the fractional initial value problem of wave type

$$\begin{aligned} {}^{C}\partial_t^{\beta_0} w(x,t) + t^{\beta_0}(-\Delta)^{\lambda} w(x,t) &= 0, \\ w(x,t)\big|_{t=0+} &= w_0(x), \\ \partial_t w(x,t)\big|_{t=0+} &= w_1(x), \end{aligned}$$

where $1 < \beta_0 \leqslant 2$ and $0 < \lambda \leqslant 1$. The solution of the above equation is

$$\begin{aligned} w(x,t) = w_0(x) - \int_{\mathbb{R}^n} \left(\mathcal{F}^{-1} I_{0+}^{\beta_0}\left(|s|^{2\lambda} t^{\beta_0} E_{1,2\beta_0,\beta_0}^{\beta_0}(-|s|^{2\lambda} t^{2\beta_0})\right)\right)(x-y) w_0(y) dy \\ + w_1(t)t - \int_{\mathbb{R}^n} \left(\mathcal{F}^{-1} I_{0+}^{\beta_0}\left(|s|^{2\lambda} t^{\beta_0+1} E_{1,2\beta_0,\beta_0+1}^{\beta_0}(-|s|^{2\lambda} t^{2\beta_0})\right)\right)(x-y) w_1(y) dy. \end{aligned}$$

Main sources

1. M. Karazym, T. Ozawa, D. Suragan. Multidimensional inverse Cauchy problems for evolution equations, Inverse Probl. Sci. En. 28(11), (2020), 1–9.
2. J.E. Restrepo, D. Suragan, "Direct and inverse Cauchy problems for generalized space-time fractional differential equations", Adv. Differential Equations, 26(7–8), (2021), 305–339. Fractional case
3. J.E. Restrepo, M. Ruzhasky, D. Suragan, "Generalized time-fractional Dirac type operators and Cauchy type problems", arXiv:2101.11725v1, 2021. Fractional case

Thank you!