Lipschitz conditions for the discrete Fourier-Laplace transform associated with the Laplace-Beltrami operator on the sphere

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Introduction

Introduction

Definition (1864)

Let $\alpha \in (0,1)$. A function $f \in L^2(\mathbb{R})$ is said in the **class Lipschitz**, denote this class by $Lip(\alpha, 2)$, if

$$\|\tau_h f(x) - f(x)\|_{L^2(\mathbb{R})} = O(h^\alpha), \quad h \longrightarrow 0. \tag{1}$$

with $\tau_h f(x) = f(x+h), x \in \mathbb{R}$.

Introduction

Theorem : Titchmarsh, 1937

If $f \in L^2(\mathbb{R})$, then the following conditions are equivalents :

$$f \in Lip(\alpha, 2) \quad , \qquad 0 < \alpha < 1,$$

$$\Updownarrow$$

$$\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda \;=\; O(r^{-2\alpha}), \quad r \to \infty.$$

Introduction

Theorem : Younis, 1974

Let $f \in L^2(\mathbb{R})$, $0 < \alpha < 1$, $\beta > 0$. Then the following conditions are equivalents :

$$\|\tau_h f(x) - f(x)\|_{L^2(\mathbb{R})} = O\left(h^\alpha \left(\log \frac{1}{h}\right)^{-\beta}\right), \quad h \to 0,$$

$$\Updownarrow$$

$$\int_{|\lambda| \geq r} |\widehat{f}(\lambda)|^2 d\lambda = O\left(r^{-2\alpha} (\log r)^{-2\beta}\right), \quad r \to \infty.$$

Introduction

- Platonov, S.S. : The Fourier transform of functions satisfying a Lipschitz condition on **symmetric spaces of rank 1**. Sibirsk. Mat. Zh. 46(6), 1374-1387 (2005)
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- El Ouadih S, Daher R. Lipschitz conditions for the generalized discrete Fourier tansform associated with the **Jacobi** operator on $[0, \pi]$. C.R.Acad.Sci.Paris,Ser.I. 2017 ; 355(3) : 318-324.
- Daher R, El Hamma M. An analog of Titchmarsh's theorem for the generalized **Dunkl** transform. J. Pseudo. Di. Op. App. 2016 ; 7(1) : 59-65.

Reminders and notations

Reminders and notations

- For more details, we can refer to [17-18].
- $\mathbb{S}^{m-1} = \left\{x = (x_1, ..., x_n) \in \mathbb{R}^m ; \sum_{i=1}^{m} |x_i|^2 = 1\right\}, \quad m \geq 3;$
- $L_2(\mathbb{S}^{m-1}) :=$
$$\left\{f \in \mathcal{M}(\mathbb{S}^{m-1}) : \|f\|_2 = \left(\frac{1}{w_m} \int_{\mathbb{S}^{m-1}} |f(x)|^2 \, d\sigma(x)\right)^{\frac{1}{2}} < \infty\right\};$$
- $w_m := \int_{\mathbb{S}^{m-1}} d\sigma = \frac{2\pi^{m/2}}{\Gamma(m/2)};$
- $(f, g) = \frac{1}{w_m} \int_{\mathbb{S}^{m-1}} f(x)g(x)d\sigma(x); f, g \in L_2(\mathbb{S}^{m-1}).$

Reminders and notations

- $\Delta_S f = (\Delta \tilde{f})_{|\mathbb{S}^{m-1}}$;
- $\Delta = \sum_{i=0}^{m} \dfrac{\partial^2}{\partial x_i^2}$;
- $\tilde{f}(x) = f\left(\dfrac{x}{\| x \|}\right)$;
- A spherical harmonic of order k is the restriction on $\mathbb{S}^{m-1}$ of a homogeneous harmonic polynomial of order k defined by the following formula

$$q(x) = \sum c_{\alpha_1,\ldots,\alpha_m} x_1^{\alpha_1} \ldots x_m^{\alpha_m}, \sum_{l=1}^{m} \alpha_l = k, \alpha_l \in \mathbb{Z}_+, \Delta q = 0.$$

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- The set of all harmonics of order k will be denoted by H_k.

Reminders and notations

- if $y_k \in H_k$, $y_l \in H_l$ and $l \neq k$, then $(y_k, y_l) = 0$;
- $L_2(\mathbb{S}^{m-1}) = \bigoplus_{k\geq 0} H_k$;
- Denote the elements of an orthonormal basis in H_k by $Y_{j,k}$ with $1 \leq j \leq d_k = \dim H_k$;
- $\Delta_S Y_{j,k} = -k(k+m-2)Y_{j,k}$;
- **Fourier-Laplace transform** : Let $f \in L_2(\mathbb{S}^{m-1})$, we have :

$$f = \sum_{k=0}^{\infty} \sum_{j=0}^{d_k} \widehat{f}_{j,k} Y_{j,k}(x) = \sum_{k=0}^{\infty} Y_k^{\lambda} f(x);$$

- where $\widehat{f}_{j,k} = (f, Y_{j,k})$;
- and Y_k^{λ} is the **orthogonal projection operator**, $Y_k^{\lambda} : L_2(\mathbb{S}^{m-1}) \longrightarrow H_k$;

Reminders and notations

- which is of the form $Y_k^\lambda f(x) = \frac{(k+\lambda)\Gamma(\lambda)}{2\pi^{\lambda+1}} \int_{\mathbb{S}^{m-1}} P_k^\lambda(x.\xi) f(\xi) d\xi(x),$
- with the $P_k^\lambda(u)$, $k = 0, 1, 2, ...,$ are the **normalized Gegenbauer polynomials** by condition $P_k^\lambda(1) = \binom{k+2\lambda-1}{k}$;
- Note also that the polynomials $P_k^\lambda(u)$, $\lambda \geq 0$, $k = 0, 1, 2, ...,$ form an orthogonal system in the closed interval $[-1, 1]$ with weight $(1-u^2)^{\lambda-1/2}$;
- **Equality of Parseval** :Let $L_2(\mathbb{S}^{m-1})$, we have :

$$\|f\|_2^2 = \sum_{k=0}^{\infty} d_k \sum_{j=1}^{d_k} |\widehat{f}_{j,k}|^2.$$

Reminders and notations

- **Spherical translation operator** for $f \in L_2(\mathbb{S}^{m-1})$:

$$S_t f(x) = \frac{1}{\omega_{m-1} \sin^{2\lambda} t} \int_{(x,y)=\cos t} f(y) d\sigma(y), \quad 0 < t < \pi.$$

- we denote by $R_k^{\lambda}(t) = \dfrac{P_k^{\lambda}(\cos t)}{P_k^{\lambda}(1)}$, we have

$$Y_k^{\lambda} S_t f(x) = R_k^{\lambda}(t) Y_k^{\lambda} f(x).$$

- Note by $\psi_r(t) = (1-t)^{r/2}$. Following the work of Rustamov [16], consider the **difference operator** defined by :

$$\Delta_t^r = (I - S_t)^{r/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \psi_r^{(n)}(0) S_t^n.$$

Reminders and notations

Proposition 1

Let $f \in L_2(\mathbb{S}^{m-1})$, then :

$$\|\Delta_t^r f\|_2 = \left(\sum_{n=0}^{\infty} \left(1 - R_k^{\lambda}(t)\right)^r \left\|Y_n^{\lambda} f(x)\right\|_2^2\right)^{1/2}.$$

Reminders and notations

Proposition 2 : Platonov 2014

The following inequalities are valid for Gegenbauer polynomials $R_k^\lambda(t)$:

1. For $t \in (0, \pi/2]$, we have $\left|R_k^\lambda(t)\right| < 1$.
2. For $t \in [0, 1]$ and $tk \leq 2$, we have
$$1 - R_k^\lambda(t) \geq C_1 k(k + 2\lambda)t^2.$$
3. For $t \in [0, \pi/2]$, we have
$$1 - R_k^\lambda(t) \leq C_2 k(k + 2\lambda)t^2.$$

Titchmarsh's theorem for **Lipschitz** functions in space $L_2(\mathbb{S}^{m-1})$

Titchmarsh's theorem for **Lipschitz** functions

Definition

Let $\alpha \in (0,1)$. A function $f \in L_2(\mathbb{S}^{m-1})$ is said in the class $(\alpha, 2)$-Fourier-Laplace Lipschitz, denote this class by $Lip(\alpha, 2)$, if

$$\|\Delta_t^1 f\|_2 = O(t^\alpha), \quad t \to 0.$$

Titchmarsh's theorem for **Lipschitz** functions

Theorem : El Ouadih, Daher 2020

Let $f \in L^2(\mathbb{S}^{m-1})$. Then the following conditions are equivalent

1. $f \in Lip(\alpha, 2)$,
2. $$\sum_{k=N}^{\infty} d_k \sum_{j=1}^{d_k} |\widehat{f}_{j,k}|^2 = O(N^{-2\alpha}), \quad as \quad N \to \infty$$

Titchmarsh's theorem for **Lipschitz** functions

For the proof of this theorem, we will use the following lemma due to **Duren** [16, p. 101].

Lemme : Duren, 1970

Suppose that $b_n \geq 0$ (the b_n are positive) and $0 < c < d$. Then

$$\sum_{j=1}^{N} j^d b_j = O(N^c) \quad as \quad N \to \infty,$$

if is only if

$$\sum_{j=N}^{\infty} b_j = O(N^{c-d}) \quad as \quad N \to \infty.$$

Titchmarsh's theorem for **Lipschitz** functions

Theorem : El Ouadih, Daher 2020

Let $f \in L_2(\mathbb{S}^{m-1})$. Then the following conditions are equivalent :

1. $\|\Delta_t^r f\|_2 = O(t^\alpha), \quad t \to 0, \quad \alpha \in (0,1), r > \alpha,$
2. $\displaystyle\sum_{k=N}^{\infty} d_k \sum_{j=1}^{d_k} |\widehat{f}_{j,k}|^2 = O(N^{-2\alpha}), \quad N \to \infty,$

Titchmarsh's theorem for **Dini-Lipschitz** functions in space $L_2(\mathbb{S}^{m-1})$

Titchmarsh's theorem for **Dini-Lipschitz** functions

Definition

Let $\beta \in \mathbb{R}$ et $\alpha \in (0,1)$. A function $f \in L_2(\mathbb{S}^{m-1})$ is said in the class $(\alpha, \beta, 2)$-Fourier-Laplace Dini-Lipschitz, denote this class by $Lip(\alpha, \beta, 2)$, if

$$\|\Delta_t^1 f(x)\|_2 = O\left(t^\alpha \left(\log t^{-1}\right)^\beta\right), \quad t \to 0.$$

Titchmarsh's theorem for **Dini-Lipschitz** functions

Theorem : El Ouadih, Daher 2020

Let $f \in L_2(\mathbb{S}^{m-1})$. Then the following conditions are equivalent :

1. $f \in Lip(\alpha, \beta, 2), \quad \alpha \in (0,1), \quad r > \alpha, \quad \beta \in \mathbb{R},$
2. $\sum_{k=N}^{\infty} d_k \sum_{j=1}^{d_k} |\widehat{f}_{j,k}|^2 = O(N^{-2\alpha}(\log N)^{2\beta}), \quad N \to \infty.$

Titchmarsh's theorem for **Dini-Lipschitz** functions

For the proof, we will use an extension of Duren lemma,

Lemme : Daher, Delgado, Ruzhansky 2019

Suppose that $a \in \mathbb{R}$, $b_n \geq 0$ and $0 < c < d$. Then

$$\sum_{n=1}^{N} n^d b_n = O(N^c (\log N)^a) \quad as \quad N \to \infty,$$

if is only if

$$\sum_{n=N}^{\infty} b_n = O(N^{c-d} (\log N)^a) \quad as \quad N \to \infty.$$

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Thank you for your attention