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Inverse Source Problems in Fractional Dual-Phase-Lag heat conduction

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13th ISAAC Congress

Session 8: Harmonic Analysis and PDEs

Ghent, August 2-6, 2021

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Overview

F. Maes, M. Slodička. Some Inverse Source Problems of Determining a Space Dependent Source in Fractional-Dual-Phase-Lag Type Equations
Mathematics (2020), 8(8), 1291

- Classical Heat Equation:

$$\rho c \partial_t T + \nabla \cdot (-\boldsymbol{k} \nabla T) = g,$$

  follows from:

  - the Fourier law

$$\boldsymbol{q}(\mathbf{x}, t) = -\boldsymbol{k}(\mathbf{x}) \nabla T(\mathbf{x}, t)$$

  - the energy balance

$$\rho c \partial_t T(\mathbf{x}, t) + \nabla \cdot \boldsymbol{q}(\mathbf{x}, t) = g(\mathbf{x}, t)$$

- Physical problem: infinite speed of propagation.
- How to resolve? Use a non-Fourier type law.

Introduction
CV - J models

- Cattaneo-Vernotte equation

$$\boldsymbol{q}(\mathbf{x}, t) + \tau \partial_t \boldsymbol{q}(\mathbf{x}, t) = -\boldsymbol{k} \nabla T(\mathbf{x}, t)$$

leads to the (hyperbolic) heat transport equation

$$\rho c \left(1 + \tau \partial_t\right) \partial_t T + \nabla \cdot \left(-\boldsymbol{k} \nabla T\right) = \left(1 + \tau \partial_t\right) g$$

where τ is the thermal relaxation time.

- Jeffreys-type heat flux equation

$$\boldsymbol{q}(\mathbf{x}, t) + \tau_q \partial_t \boldsymbol{q}(\mathbf{x}, t) = -\boldsymbol{k}(\mathbf{x}) \left(\nabla T(\mathbf{x}, t) + \tau_T \partial_t \nabla T(\mathbf{x}, t)\right)$$

leads to

$$\rho c \left(1 + \tau_q \partial_t\right) \partial_t T + \nabla \cdot \left(-\boldsymbol{k} \left(1 + \tau_T \partial_t\right) \nabla T\right) = \left(1 + \tau_q \partial_t\right) g$$

where τ_q and τ_T are delay times.

Introduction

Tzou's model

- Dual-Phase-Lag model

$$\boldsymbol{q}(\mathbf{x}, t + \tau_q) = -\boldsymbol{k}\nabla T(\mathbf{x}, t + \tau_T)$$

- Interpretation:
  - τ_T caused by micro-structural interactions
  - τ_q relaxation time due to fast transient effects of thermal inertia
- Generalizes CV and J type equations
- Gives rise to new models by Taylor approximation, e.g.

$$\left(1 + \tau_q\partial_t + \frac{\tau_q^2}{2}\partial_{tt}\right)\boldsymbol{q}(\mathbf{x}, t) = -\boldsymbol{k}\left(1 + \tau_T\partial_t\right)\nabla T(\mathbf{x}, t).$$

D. Tzou. A Unified Field Approach for Heat Conduction From Macro- to Micro-Scales
J. Heat Transf., 1995, 117, 8-16

D.S. Chandrasekharaiah. Hyperbolic thermoelasticity: A review of recent literature
Appl. Mech. Rev., 51 (12), 705-729, 1998

R. Quintanilla, R. Racke. Qualitative aspects in dual-phase-lag heat conduction
Proc. R. Soc. A (2007) 463, 659-674

- Fractional version

$$\left(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha\right)\boldsymbol{q}(\mathbf{x},t)=-\boldsymbol{k}(\mathbf{x})\left(1+\tau_T^\beta\,\mathrm{D}_t^\beta\right)\nabla T(\mathbf{x},t),$$

where D_t^α is the Caputo fractional derivative

$$\left({}_a^C\mathrm{D}_t^\alpha f\right)(t):=\left({}_a\mathrm{I}_t^{n-\alpha}\,\mathrm{D}_t^n f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-1-\alpha}f^{(n)}(\tau)\,\mathrm{d}\tau$$

for $n-1\le\alpha<n$, and ${}_a^C\mathrm{D}_t^\alpha=\mathrm{D}_t^n$ for $\alpha=n$.

- Fractional Dual-Phase-Lag Heat Equation $(0<\alpha,\beta<1)$

$$\left(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha\right)\left(\rho c\partial_t T(\mathbf{x},t)\right)+\nabla\cdot\left[-\boldsymbol{k}(\mathbf{x})\left(1+\tau_T^\beta\,\mathrm{D}_t^\beta\right)\nabla T(\mathbf{x},t)\right]=(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha)g(\mathbf{x},t).$$

- In some practical situations (e.g. bioheat, thermal therapy of skin tissue),

$$\left(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha\right)g(\mathbf{x},t)=F(\mathbf{x})+f(\mathbf{x})h(t)-a\left(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha\right)T(\mathbf{x},t),\quad a\ge 0.$$

J. Singh, P.K. Gupta, K.N. Rai. Solution of fractional bioheat equations by finite difference method and HPM *Math. Comput. Model. 54 (2011) 2316-2325)*

Introduction
FDPL: final form

- Bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$ with $\partial\Omega$ Lipschitz continuous.
- Final time $\mathcal{T} > 0$.

For $(\mathbf{x}, t) \in \Omega \times (0, \mathcal{T}]$ consider

$$\begin{aligned}\left(1 + \tau_q^\alpha \, \mathrm{D}_t^\alpha\right)\left(\rho c \partial_t T(\mathbf{x}, t)\right) + a\left(1 + \tau_q^\alpha \, \mathrm{D}_t^\alpha\right) T(\mathbf{x}, t) \\ + \nabla \cdot \left[-\boldsymbol{k}(\mathbf{x})\left(1 + \tau_T^\beta \, \mathrm{D}_t^\beta\right)\nabla T(\mathbf{x}, t)\right] = F(\mathbf{x}) + f(\mathbf{x})h(t) \qquad (1)\end{aligned}$$

with initial data

$$T(\mathbf{x}, 0) = T_0(\mathbf{x}) \quad \text{and} \quad \partial_t T(\mathbf{x}, 0) = V_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad (2)$$

and boundary condition

$$T(\mathbf{x}, t) = T_\Gamma(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \partial\Omega \times (0, T]. \qquad (3)$$

Goal: Study uniqueness of a solution $\left(T(\mathbf{x}, t), f(\mathbf{x})\right)$ to the ISP, given final time observation $T(\mathbf{x}, \mathcal{T}), \mathbf{x} \in \Omega$.

Preliminaries

- Caputo Derivative

$$\left(\mathrm{D}_t^{\alpha} f\right)(t) = \left(g_{1-\alpha} * \partial_t f\right)(t) = \int_0^t g_{1-\alpha}(t-\tau)\partial_t f(\tau)\,\mathrm{d}\tau.$$

- Riemann-Liouville kernel $g_{1-\alpha} \in \mathrm{L}^1_{\mathrm{loc}}(0,\infty)$

$$g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad t > 0, \quad 0 < \alpha < 1,$$

satisfies $g'_{1-\alpha} \not\equiv 0$ and is strongly positive definite, i.e.

$$\int_0^t \left(g_{1-\alpha} * y\right)(s) y(s) \mathrm{d}s \geq 0$$

Lemma

Let $0 < \alpha < 1$ *and* $\partial_t v \in \mathrm{L}^2_{loc}\left((0,\infty), \mathrm{L}^2(\Omega)\right)$ *with* $v(0) = 0$. *Then for any* $\xi \geq 0$, *we have*

$$\int_0^{\xi} \left((\mathrm{D}_t^{\alpha} v)(s), v(s)\right) \mathrm{d}s \geq 0.$$

Preliminaries

Lemma

Let $0 < \beta < 1$ *and* $\mathcal{T} > 0$. *Consider functions* $v, r, z \in \mathrm{C}\left([0, \mathcal{T}]\right)$, *and* $\alpha \in \mathrm{C}^1\left((0, \mathcal{T}]\right)$ *with* $\alpha' \in \mathrm{L}^1(0, \mathcal{T})$. *Assume that* α *obeys*

$$z(t)\alpha'(t) + \mathrm{D}^{\beta}\,\alpha(t) + r(t)\alpha(t) = v(t) \quad t \in [0, \mathcal{T}],$$

along with

$$\alpha(0) = \alpha(\mathcal{T}) = 0, \quad 0 \leq r(t), z(t), \quad t \in [0, \mathcal{T}].$$

If $v(t) \leq 0$ *for* $t \in [0, \mathcal{T}]$, *then* $\alpha(t) \leq 0$ *for* $t \in [0, \mathcal{T}]$.

Preliminaries

Let $\lambda > 0$ and consider

$$\mathrm{D}^{\beta}\, v(t) + \lambda v(t) = \sigma(t), \quad v(0) = 0.$$

The solution is given by

$$v(t) = \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\left(-\lambda(t-s)^{\beta}\right)\sigma(s)\,\mathrm{d}s,$$

where

$$E_{\mu,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \gamma)}$$

is the Mittag-Leffler function. Note that for $\mu, \gamma > 0$, the function $E_{\mu,\gamma}(-t)$, $t > 0$, is completely monotonic iff. $0 < \mu \leq 1$, and $\mu \leq \gamma$.

- I. Podlubný. Fractional Differential Equations (1998)
- T.M. Atanacković, S. Pilipović, B. Stanković, D. Zorica. Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes. (2014)
- F. Mainardi, R. Gorenflo. Time-fractional derivatives in relaxation processes: A tutorial survey *Fract. Calc. Appl. Anal., 10 (3), 2007*
- K.S. Miller, S.G. Samko A note on the complete monotonicity of the gernalized Mittag-Leffler function *Real Anal. Exch., (23) 753-756, 1999*

Main result
Problem description

Consider the homogeneous problem (1)-(2)-(3):

$$
\left\{
\begin{array}{rll}
\left(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha\right)\left(\rho c \partial_t T(\mathbf{x},t)\right) + a\left(1+\tau_q^\alpha\,\mathrm{D}_t^\alpha\right) T(\mathbf{x},t) & & \\
+\nabla\cdot\left[-\boldsymbol{k}(\mathbf{x})\left(1+\tau_T^\beta\,\mathrm{D}_t^\beta\right)\nabla T(\mathbf{x},t)\right] & = f(\mathbf{x})h(t) & \text{in } \Omega\times(0,\mathcal{T}] \\
T(\mathbf{x},t) & = 0 & \text{in } \partial\Omega\times(0,\mathcal{T}] \\
T(\mathbf{x},0) & = 0 & \text{in } \Omega \\
\partial_t T(\mathbf{x},0) & = 0 & \text{in } \Omega \\
T(\mathbf{x},\mathcal{T}) & = 0 & \text{in } \Omega
\end{array}
\right. \tag{4}
$$

Goal: $T(\mathbf{x},t) = 0 = f(\mathbf{x})$ a.e.

Main result

Theorem

Let $0 < \alpha, \beta < 1$. *Assume* $\boldsymbol{k}(\mathbf{x})$ *satisfies,*

$$\begin{cases} \boldsymbol{k}(\mathbf{x}) & = \left(k_{i,j}(\mathbf{x})\right)_{i,j=1,\ldots,d}, \boldsymbol{k} = \boldsymbol{k}^\top, \quad \mathbf{x} \in \Omega, \\ k_{i,j} & \in \mathrm{L}^\infty(\Omega) \qquad\qquad\qquad\qquad i,j = 1,\ldots,d, \\ \boldsymbol{\xi}^\top \cdot \boldsymbol{k}\boldsymbol{\xi} & = \sum_{i,j=1}^{d} k_{i,j}(\mathbf{x})\xi_i\xi_j \geq \underline{k}\,|\boldsymbol{\xi}|^2, \qquad \underline{k} > 0, \forall \boldsymbol{\xi} \in \mathbb{R}^d. \end{cases}$$

Let $\rho, c > 0$ *and* $a, \tau_q^\alpha, \tau_T^\beta \geq 0$. *Moreover suppose that* $h \in \mathrm{C}\left([0, \mathcal{T}]\right)$ *and* $h' \in \mathrm{C}\left((0, \mathcal{T}]\right)$ *with*

$$h(t) > 0, \quad h'(t) \geq 0.$$

If (T, f) *is a solution to* (4) *with*

$$T \in \mathrm{C}\left([0, \mathcal{T}], \mathrm{H}_0^1(\Omega)\right), \quad \partial_t T \in \mathrm{L}^2\left((0, \mathcal{T}), \mathrm{H}_0^1(\Omega)\right), \quad f \in \mathrm{L}^2(\Omega),$$

then $T(\mathbf{x}, t) = 0 = f(\mathbf{x})$ *a.e.*

Main result
Sketch of the proof

- Proof is based on the variational technique.
- The crucial observation is here

$$\int_0^{\mathcal{T}} (f, \partial_t T(s))\,\mathrm{d}s = (f, T(\mathcal{T}) - T(0)) = 0.$$

- Let $H(t) = 1/h(t)$. Then we have

$$\rho c \int_0^{\mathcal{T}} H(s)\,\|\partial_t T(s)\|^2\,\mathrm{d}s + \rho c \tau_q^\alpha \int_0^{\mathcal{T}} H(s)\,\left(\mathrm{D}_t^{\alpha+1}\,T(s), \partial_t T(s)\right)\mathrm{d}s$$
$$+ \int_0^{\mathcal{T}} H(s)\,(aT(s), \partial_t T(s))\,\mathrm{d}s + \tau_q^\alpha \int_0^{\mathcal{T}} H(s)\,(a\,\mathrm{D}_t^\alpha\,T(s), \partial_t T(s))\,\mathrm{d}s$$
$$+ \int_0^{\mathcal{T}} H(s)\,(\boldsymbol{k}\nabla T(s), \nabla\partial_t T(s))\,\mathrm{d}s + \tau_T^\beta \int_0^{\mathcal{T}} H(s)\,\left(\boldsymbol{k}\,\mathrm{D}_t^\beta\,\nabla T(s), \nabla\partial_t T(s)\right)\mathrm{d}s = 0.$$

- Standard estimation procedure yields

$$\int_0^{\mathcal{T}} H(s)\,\|\partial_t T(s)\|^2\,\mathrm{d}s = 0.$$

Main result
Remark

- Conclusion is false if h, h' change sign.

Example

Let $\Omega = (0, \pi)$ and $\mathcal{T} = 1$. Consider

$$\begin{cases} \left(1 + \tau_q^\alpha \, \mathrm{D}_t^\alpha\right) \left(\partial_t T(x,t)\right) - \left(1 + \tau_T^\beta \, \mathrm{D}_t^\beta\right) \Delta T(x,t) & = f(x)h(t) \quad \text{in } \Omega \times (0, \mathcal{T}] \\ T(x,t) & = 0 \quad \text{in } \partial\Omega \times (0, \mathcal{T}] \\ T(x,0) = T(x,\mathcal{T}) & = 0 \quad \text{in } \Omega. \end{cases}$$

Set $T(x,t) = v(t)g(x)$, and $f(x) = g(x)$ with $-\Delta g = \lambda g$, $g(0) = g(\pi) = 0$, and $v \in \mathrm{C}^2(0,1)$ with $v(0) = v(1)$. Then

$$h(t) = \tau_q^\alpha \left(g_{1-\alpha} * v''\right)(t) + \lambda \tau_T^\beta \left(g_{1-\beta} * v'\right)(t) + v'(t) + \lambda v(t)$$

Modified models

Model 1: $\tau_T > \tau_q$

- Let $\tau := \tau_T - \tau_q > 0$ and consider

$$\boldsymbol{q}(\mathbf{x}, t) = -\boldsymbol{k}(\mathbf{x})\nabla T(\mathbf{x}, t + \tau).$$

- First order Taylor expansion and the fractional version yields

$$\rho c \partial_t T(\mathbf{x}, t) + \nabla \cdot \left[-\boldsymbol{k}(x) \left(1 + \tau^\beta \, \mathrm{D}_t^\beta \right) \nabla T(\mathbf{x}, t) \right] = g(\mathbf{x}, t)$$

- Take $g(\mathbf{x}, t) = F(\mathbf{x}) + f(\mathbf{x})h(t) - aT(\mathbf{x}, t)$.
- Behavior is dominated by the parabolic terms.
- ISP: Find $T(\mathbf{x}, t)$ and $f(\mathbf{x})$ obeying

$$\begin{cases} \rho c \partial_t T(\mathbf{x}, t) + aT(\mathbf{x}, t) & \\ \quad + \nabla \cdot \left[-\boldsymbol{k}(\mathbf{x}) \left(1 + \tau^\beta \, \mathrm{D}_t^\beta \right) \nabla T(\mathbf{x}, t) \right] = F(\mathbf{x}) + f(\mathbf{x})h(t) & \text{in } \Omega \times (0, \mathcal{T}] \\ T(\mathbf{x}, t) = T_\Gamma(\mathbf{x}, t) & \text{in } \partial\Omega \times (0, \mathcal{T}] \\ T(\mathbf{x}, 0) = T_0(\mathbf{x}) & \text{in } \Omega \\ T(\mathbf{x}, \mathcal{T}) = \Psi(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{5}$$

Y. Luchko Maximum Principle and Its Application for the Time-Fractional Diffusion Equations *Fract. Calc. Appl. Anal. (2011), 14, 110-124*

Modified models
Model 1: $\tau_T > \tau_q$

Theorem

Let $0 < \beta < 1$, *assume the same conditions on* $\boldsymbol{k}$ *and the coefficients as before. Suppose now that* $h \in \mathrm{C}\left((0, \mathcal{T}]\right)$, *with* $h \geq 0$ *and* $h \not\equiv 0$. *Then there exists at most one couple* $\left(T(\mathbf{x}, t), f(\mathbf{x})\right)$ *solving the ISP* (5) *such that*

$$T \in \mathrm{C}\left([0, \mathcal{T}], \mathrm{L}^2(\Omega)\right), \quad T \in \mathrm{L}^2\left((0, \mathcal{T}), \mathrm{H}_0^1(\Omega)\right), \quad f \in \mathrm{L}^2(\Omega),$$

and for some $0 < \delta < 1$,

$$|h(t)| + \|T_t(t)\| \leq C\left(1 + t^{-\delta}\right) \quad \text{for any } t \in (0, \mathcal{T}].$$

- Technique is by separation of variables and spectral analysis of $Av := \nabla \cdot [-\boldsymbol{k} \nabla v]$ acting on $\mathrm{H}_0^1(\Omega)$.

Modified models
Model 1: $\tau_T > \tau_q$ – Sketch

- Consider the homogeneous problem.
- Let $e_j \in \mathrm{H}_0^1(\Omega)$ be an orthonormal base in $\mathrm{L}^2(\Omega)$ of eigenfunctions of A with $Ae_j = \lambda_j e_j$. Here $0 < \lambda_j \to \infty$.
- Let

$$T(t) = \sum_{j=0}^{\infty} \alpha_j(t) e_j \quad \text{and} \quad f = \sum_{j=0}^{\infty} (f, e_j)\, e_j$$

  then $\alpha_j(0) = 0 = \alpha_j(\mathcal{T})$ for all j and w.l.o.g. $(f, e_j) \geq 0$.
- The governing fractional partial differential equation then gives

$$\frac{\rho c}{\tau^\beta \lambda_j} \alpha_j'(t) + \frac{a + \lambda_j}{\tau^\beta \lambda_j} \alpha_j(t) + \left(\mathrm{D}_t^\beta \, \alpha_j \right)(t) = \frac{h(t)}{\tau^\beta \lambda_j} (f, e_j) \quad \text{for all } j$$

- By the lemma $\alpha_j(t) \geq 0$ for all j.

Modified models
Model 1: $\tau_T > \tau_q$ – Sketch

Governing fractional partial differential equation, $\alpha_j(t) \geq 0$,

$$\frac{\rho c}{\tau^\beta \lambda_j}\alpha_j'(t) + \frac{a+\lambda_j}{\tau^\beta \lambda_j}\alpha_j(t) + \left(\mathrm{D}_t^\beta\, \alpha_j\right)(t) = \frac{h(t)}{\tau^\beta \lambda_j}\,(f, e_j) \quad \text{for all } j$$

- Here, we can express $\alpha_j(t)$ in terms of $E_{\beta,\beta}$.

$$\begin{aligned}\alpha_j(t) \quad &= \frac{(f, e_j)}{\tau^\beta \lambda_j}\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\left(-\frac{a+\lambda_j}{\tau^\beta \lambda_j}(t-s)^\beta\right) h(s)\,\mathrm{d}s \\ &\quad -\frac{\rho c}{\tau^\beta \lambda_j}\int_0^t (t-s)^{\beta-1} E_{\beta,\beta}\left(-\frac{a+\lambda_j}{\tau^\beta \lambda_j}(t-s)^\beta\right) \alpha_j'(s)\,\mathrm{d}s\end{aligned}$$

- Using several monotonicity properties of the Mittag-Leffler function we find

$$\alpha_j(s) = 0 = h(s)\,(f, e_j) \quad \text{in } (0, \mathcal{T}].$$

Modified models
Model 2: $\tau_T < \tau_q$

- Let $\tau := \tau_q - \tau_T > 0$ and consider (single-phase-lag)

$$\boldsymbol{q}(\mathbf{x}, t + \tau) = -\boldsymbol{k}(\mathbf{x}) \nabla T(\mathbf{x}, t)$$

- First order Taylor expansion and fractional version yields

$$\left(1 + \tau^{\alpha}\, \mathrm{D}_t^{\alpha}\right) \left(\rho c \partial_t T(\mathbf{x}, t)\right) + \nabla \cdot \left[-\boldsymbol{k}(\mathbf{x}) \nabla T(\mathbf{x}, t)\right] = \left(1 + \tau^{\alpha}\, \mathrm{D}_t^{\alpha}\right) g(\mathbf{x}, t).$$

- Take $\left(1 + \tau^{\alpha}\, \mathrm{D}_t^{\alpha}\right) g(\mathbf{x}, t) = F(\mathbf{x}) + f(\mathbf{x}) h(t) - a \left(1 + \tau^{\alpha}\, \mathrm{D}_t^{\alpha}\right) T(\mathbf{x}, t)$.
- Fractional Cattaneo-Vernotte type equation
- The result as in Model 1, cannot be proved for this system.
- Example showing positivity of h is not sufficient for uniqueness.

Modified models
Model 2: $\tau_T < \tau_q$

Example

Let $\mathcal{T} = 2\pi$ and consider

$$\begin{cases} (1 + \mathrm{D}_t^{\alpha})\, \partial_t T(\mathbf{x}, t) - \Delta T(\mathbf{x}, t) & = f(\mathbf{x}) h(t) & \text{in } \Omega \times (0, \mathcal{T}] \\ T(\mathbf{x}, 0) & = 0 & \mathbf{x} \in \Omega \\ T_t(\mathbf{x}, 0) & = 0 & \mathbf{x} \in \Omega \\ T(\mathbf{x}, t) & = 0 & (\mathbf{x}, t) \in \partial\Omega \times (0, \mathcal{T}] \end{cases}$$

with final time measurement $T(\mathbf{x}, \mathcal{T}) = 0$.

- The ISP for determining the missing $f(\mathbf{x})$ has the trivial solution $(T(\mathbf{x}, t), f(\mathbf{x})) = (0, 0)$.
- Construction of a non-trivial solution ($\alpha = \frac{1}{2}$):
  1. set $f = e$ with $-\Delta e = \lambda e$ with $e \in \mathrm{H}_0^1(\Omega)$
  2. set $T(\mathbf{x}, t) = v(t) e(\mathbf{x})$
  3. $v'(t) + \left(D_t^{\alpha} v'\right) + \lambda v(t) = h(t), \quad v(0) = 0 = v'(0), \quad v(\mathcal{T}) = 0$
  4. consider $v(t) = 1 - \cos(t)$
  5. determine h

Modified models
Model 2: $\tau_T < \tau_q$

Example

Consider

$$\begin{cases} (1 + \mathrm{D}_t^\alpha)\,\partial_t T(\mathbf{x},t) - \Delta T(\mathbf{x},t) &= f(\mathbf{x})h(t) & \text{in } \Omega \times (0,\mathcal{T}] \\ T(\mathbf{x},0) &= 0 & \mathbf{x} \in \Omega \\ T_t(\mathbf{x},0) &= 0 & \mathbf{x} \in \Omega \\ T(\mathbf{x},t) &= 0 & (\mathbf{x},t) \in \partial\Omega \times (0,\mathcal{T}] \end{cases}$$

with final time measurement $T(\mathbf{x},\mathcal{T}) = 0$.

- Construction of a non-trivial solution:
  5. determine h,

  $$h(t) = \sin(t) + \lambda(1-\cos(t)) + \sqrt{2}\left[\cos(t)\,\mathcal{C}\left(\sqrt{\frac{2t}{\pi}}\right) + \sin(t)\,\mathcal{S}\left(\sqrt{\frac{2t}{\pi}}\right)\right]$$

  choose $\lambda > 0$ sufficiently large. Then $h \geq 0$, $h \not\equiv 0$ and h' changes sign.
  6. $(T(\mathbf{x},t), f(\mathbf{x})) = ((1-\cos(t))e(\mathbf{x}), e(\mathbf{x}))$ is a non-trivial solution.

Thank you!