Asymptotic behavior for the nonlinear damped wave equation with oscillating coefficients

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CONTENTS

Physical Backgroud In quantum mechanics and electrodynamics, we frequently confront many nonlinear hyperbolic problems with time dependent coefficients. Compared with the cases of constant coefficients and spacial coefficients, the time-dependent coefficients not only influence the well-posedness of the solutions, but they also pose great challenges to the conservation laws obeyed by the hyperbolic systems.

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Nonlinear damped wave model Let $\Omega \in \mathbb{R}^N$ be a bounded Lipschitz domain with the smooth boundary Γ. In this talk, we focus on the nonlinear damped wave equation arising from the research of ultrasonic propagation in human tissues. Let us define the second order operator

$$\Box_A := \partial_t^2 + \partial_t - A^2(t)\Delta.$$

$$\begin{cases} \Box_A u = \dfrac{\eta F''(u)(A^2(t)|\nabla u|^2 - u_t^2)}{1+\eta F'(u)}, & \text{in } (0,+\infty)\times\Omega, \\ \\ u(t,x) = -\eta F(u(t,x)), & \text{on } \Gamma, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & \text{on } \Omega, \end{cases}$$

provided that the nonlinear function $F \in C^2(\mathbb{R})$, $\eta \geq 0$ and $1 + \eta F'(x) > 0$ for $\forall\ x \in \mathbb{R}$.

- $\eta = 0$ or $F(x) = x$, this corresponds to the linear damped wave equation with Dirichlet boundary condition.
- $F(x) \equiv \mathrm{Cons}$, this corresponds to the linear damped wave equation with non-Dirichlet boundary condition.

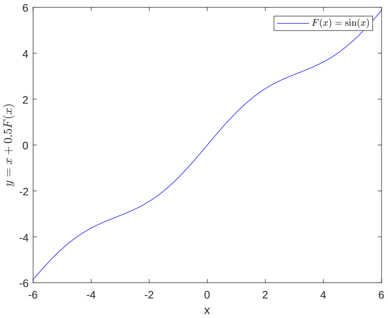

Figure: Trigonometric case $F(x) = \sin(x)$

$$\begin{cases} \Box_A u = \dfrac{-0.5 \sin u (A^2(t)|\nabla u|^2 - u_t^2)}{1 + 0.5 \cos u}, & \text{in } (0, +\infty) \times \Omega, \\ u(t,x) = -0.5 \sin(u(t,x)), & \text{on } \Gamma, \\ u(0,x) = u_0(x), \; u_t(0,x) = u_1(x), & \text{on } \Omega. \end{cases}$$

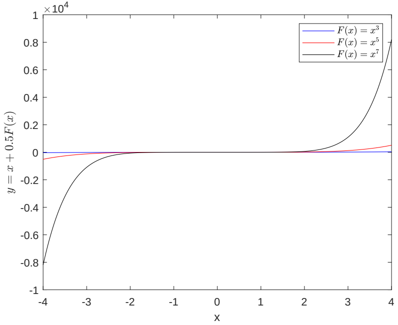

Figure: Polynomial case $F(x) = x^n$, n is odd

$$\begin{cases} \square_A u = \dfrac{0.5n(n-1)u^{n-2}(A^2(t)|\nabla u|^2 - u_t^2)}{1 + 0.5nu^{n-1}}, & \text{in}(0, +\infty) \times \Omega, \\ u(t, x) = 0, & \text{on } \Gamma, \\ u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & \text{on } \Omega. \end{cases}$$

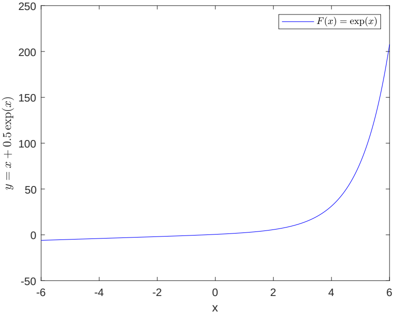

Figure: Exponential case $F(x) = \exp(x)$

$$\begin{cases} \Box_A u = \dfrac{0.5\exp(u)(A^2(t)|\nabla u|^2 - u_t^2)}{1 + 0.5\exp(u)}, & \text{in } (0, +\infty) \times \Omega, \\ u(t,x) = -0.5\exp(u(t,x)), & \text{on } \Gamma, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), & \text{on } \Omega. \end{cases}$$

Spectral theory For the bounded Lipschitz domain with the smooth boundary Γ, the Dirichlet Laplacian $-\Delta$ is a self-adjoint, positive definite operator with compact resolvent whose spectrum set

$$\mathrm{Spec}(-\Delta) = \{\lambda_i^2\}_{i\in\mathbb{N}_+}$$

is composed of discrete points with finite multiplicity, which are arranged in order as

$$0 < \lambda_1^2 \leq \lambda_2^2 \leq \lambda_3^2 \leq \cdots \to +\infty.$$

Moreover, there exists a complete orthonormal basis composed of eigenfunctions as $\{\phi_\lambda(x)\}_{\lambda^2\in\mathrm{Spec}(-\Delta)}$ in $L^2(\Omega)$, that is to say, $\|\phi_\lambda\|_{L^2} = 1$ for each $\lambda^2 \in \mathrm{Spec}(-\Delta)$.

The electrons in an atom are arranged in energy levels (shells) around the nucleus. The lowest energy level, called the first energy level or first shell, is the one closest to the nucleus. The shells increase in energy as they get further from the nucleus.

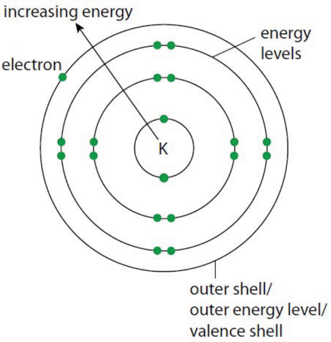

Figure: The electron arrangement of potassium K

Each line in the spectrum comes from the transition of an electron from a high energy level to a lower one. Electrons in an atom are allowed to have only certain amounts of energy, while electrons outside the atom can have any energy.

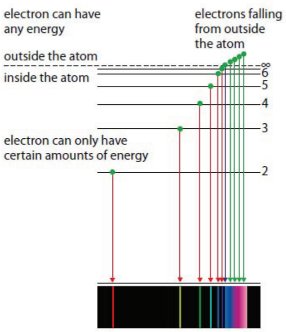

Figure: Representation of the Lyman series of hydrogen in the ultraviolet region of the electromagnetic spectrum

- **Assumption 1:** The function $A : [0, +\infty) \to \mathbb{R}$ is an oscillating coefficient on the principal Laplacian operator as time tends to infinity.
- **Assumption 2:** The variable coefficient A is uniformly bounded, that is to say, there exist two positive constants b_0 and b_1 such that

$$0 < b_0 = \min_{t \in [0,+\infty)} A(t) \leq A(t) \leq \max_{t \in [0,+\infty)} A(t) = b_1 < +\infty.$$

- **Assumption 3:** In order to exclude the case of constant coefficient on the Laplacian part, it is reasonable to assume that, for any $\lambda^2 \in \mathrm{Spec}(-\Delta)$,

$$A(t)\lambda \neq 0.5.$$

- **Assumption 4:** Let the positive function $\nu \in C^2([0,+\infty))$ be either a strictly increasing function or a constant. Furthermore, assume that, for $\forall\ \delta < 0$, there exists an $M_1 \geq 0$, such that, $((\beta+t)^\delta \nu^k(t))^{-1}$ is strictly increasing as $t > M_1$, and

$$\lim_{t\to+\infty} (\beta+t)^\delta \nu^k(t) = 0, \ \ k = 1,2.$$

- **Assumption 5:** Let us assume that $A \in C^2([0,+\infty))$ and there exist nonnegative constants C_k, $k = 1,2$, such that the multiple derivatives of $A(t)$ satisfy

$$|A^{(k)}(t)| \leq C_k (\beta+t)^\alpha \left(\frac{\nu(t)}{\beta+t}\right)^k, k = 1,2, \ \ \alpha < 0;$$

$$|A^{(k)}(t)| \leq C_k \Big((\beta+t)^{\alpha-1}\nu(t)\Big)^k, k = 1,2, \ \ \alpha \geq 0.$$

Oscillation classification Indeed, the oscillation of $A(t)$ can be classified into four groups:

- $\alpha \geq 1$, strong oscillation;
- $0.5 \leq \alpha < 1$, regular oscillation;
- $0 \leq \alpha < 0.5$, mild oscillation;
- $\alpha < 0$, weak oscillation.

For instance, one can easily check that, $A(t) = 2 + \sin((\beta + t)^{\alpha}(\ln(\beta + t))^{\kappa-1})$, $\alpha \in \mathbb{R}$, $\kappa \geq 1$ is a typical example with $\nu(t) = (\ln(\beta + t))^{\kappa-1}$ in this respect.

Definition

For the sufficiently smooth(with regards to the time) function $w : [0, +\infty) \times \Omega \to \mathbb{R}$ belonging to the function space $\mathscr{U}$,

$$\mathscr{U} := \left\{ w \,\middle|\, (1+\eta F'(w))|\nabla w|, (1+\eta F'(w))w_t \in C([0, +\infty); L^2(\Omega)) \right\},$$

we introduce the weighted nonhomogeneous energy,

$$\mathbf{E}_\eta(w)(t) := A^2(t)\|(1+\eta F'(w))|\nabla w(t, \cdot)|\|_{L^2}^2 + \|(1+\eta F'(w))w_t(t, \cdot)\|_{L^2}^2.$$

It is evident that if $\eta = 0$, then, the above energy degenerates into the classical energy form for wave equations. In addition, for the nonlinear term F satisfying $F' \geq 0$, $w \in \mathscr{U}$ indicates that for any t, $w(t, \cdot) \in H^1(\Omega)$.

As we recall the damped wave operator with spacial coefficient

$$\Box_1 = \partial_t^2 + \partial_t - A^2(x)\Delta,$$

if $A(x)$ satisfies

$$0 < d_0 < \inf_{x\in\Omega} A(x) \leq A(x) \leq \sup_{x\in\Omega} A(x) < d_1 < +\infty,$$

then, in comparison with the classical damped wave operator

$$\Box_0 = \partial_t^2 + \partial_t - \Delta,$$

the energy decays almost similarly with minor modifications for the corresponding linear Cauchy problems. However, it is remarkable to observe that, once the coefficient oscillates with respect to time, the energy decay or growth will be impacted in a substantial way.

Theorem (Energy estimates)

Let us consider the nonlinear damped wave equation. Let λ_1^2 *be the first eigenvalue of the Dirichlet Laplacian* $-\Delta$*. Provided that the initial Cauchy data satisfy*

$$(1+\eta F'(u_0))|\nabla u_0| \in L^2(\Omega), \quad (1+\eta F'(u_0))u_1 \in L^2(\Omega).$$

Then, there exists a unique solution u *belonging to the function space* $\mathscr{U}$ *with the following asymptotic energy decay rate.*

Theorem (Energy estimates)

Case I: *If there exist a finite number of* $\lambda^2 \in \mathrm{Spec}(-\Delta)$ *such that* $A(t)\lambda < 0.5$*, we have the following estimates.*

- *For the weak, mild and regular oscillations* $\alpha < 1$*, we have*

$$\mathbf{E}_\eta(u)(t) \leq C_1 \mathrm{e}^{(-1+2\sqrt{0.25-b_0^2\lambda_1^2})t} \mathbf{E}_\eta(u)(0).$$

- *For the case of strong oscillations* $\alpha = 1$*, we have*

$$\mathbf{E}_\eta(u)(t) \leq C_1 \mathrm{e}^{C_2(\beta+t)\nu(t)} \mathbf{E}_\eta(u)(0).$$

Theorem (Energy estimates)

Case II: *If for any* $\lambda^2 \in \mathrm{Spec}(-\Delta)$, *it holds that,* $A(t)\lambda > 0.5$, *we have the following estimates.*

- *For the mild oscillation* $\alpha < 0.5$, *we have* $\mathbf{E}_\eta(u)(t) \leq C_1 \mathrm{e}^{-t} \mathbf{E}_\eta(u)(0)$.
- *For the special case of regular oscillations* $\alpha = 0.5$, *we have* $\mathbf{E}_\eta(u)(t) \leq C_1 \mathrm{e}^{-t} \mathrm{e}^{C_2 \ln(\beta+t)\nu^2(t)} \mathbf{E}_\eta(u)(0)$.
- *For the general case of regular oscillations* $0.5 < \alpha < 1$, *we have* $\mathbf{E}_\eta(u)(t) \leq C_1 \mathrm{e}^{-t} \mathrm{e}^{C_2 (\beta+t)^{2\alpha-1}\nu^2(t)} \mathbf{E}_\eta(u)(0)$.
- *For the case of strong oscillations* $\alpha = 1$, *we have* $\mathbf{E}_\eta(u)(t) \leq C_1 \mathrm{e}^{C_2 (\beta+t)\nu(t)} \mathbf{E}_\eta(u)(0)$.

Here, C_1 *and* C_2 *are both positive constants.*

It is evident that, low frequencies play essential roles in the determination of energy decay estimates. In contrast to the linear wave equations, it is found out that, the linear damping term u_t contributes a decay rate e^{-t}. It is evident that, for $\alpha < 1$, generally speaking, the energy decays exponentially. These estimates are much more delicate than the conclusion reached by simply applying the energy method. We observe that, for $\alpha \geq 1$, at most we have an exponential type growth rate for the nonlinear damped wave equation. Especially, $\alpha = 1$ serves as a critical case between the energy decay and growth. Whether the energy really grows exponentially or not, is what we will focus uopn in the next step.

Theorem (Optimality estimates)

For the critical case of $\alpha = 1$ and $\nu \equiv 1$, there exists

- *a quasi-periodic coefficient $A^2(t)$ satisfying Assumptions 1-5;*
- *nontrivial initial Cauchy data satisfying*

$$(1+\eta F'(u_0))|\nabla u_0| \in L^2(\Omega), \quad (1+\eta F'(u_0))u_1 \in L^2(\Omega);$$

- *a sequence of time $\{t_n\}_n$ which tends to $+\infty$,*

such that the weighted nonhomogeneous energy of the corresponding solution $u \in \mathscr{U}$ of the nonlinear damped wave equation grows exponentially from below, that is to say, we have

$$\mathbf{E}_\eta(u)(0) \leq 50\pi \mathrm{e}^{2\beta},$$

$$\lim_{n\to\infty} \mathrm{e}^{-c(\beta+t_n)} \mathbf{E}_\eta(u)(t_n) = +\infty,$$

where β is a sufficiently large positive real number and $0 < c < 1$.

Sketch of proof of energy estimates

Definition

- For $\alpha < 0$, we divide the quantized time-frequency space into two zones $\mathbb{Z}_i$, $i = 1, 2$, namely,

$$\mathbb{Z}_1 := \{(t, \lambda^2) : (\beta + t)\lambda \in (0, 2^P)\};$$
$$\mathbb{Z}_2 := \{(t, \lambda^2) : (\beta + t)\lambda \in [2^P, +\infty)\}.$$

- For $0 \le \alpha \le 1$, we divide the quantized time-frequency space into two zones $\mathbb{Z}_i$, $i = 1, 2$, namely,

$$\mathbb{Z}_1 := \{(t, \lambda^2) : (\beta + t)\lambda \in (0, 2^P(\beta + t)^\alpha \nu(t))\};$$
$$\mathbb{Z}_2 := \{(t, \lambda^2) : (\beta + t)\lambda \in [2^P(\beta + t)^\alpha \nu(t), +\infty)\}.$$

Here, $P \in \mathbb{N}_+$ is a sufficiently large and positive constant.

Definition

Here, t_λ is defined as the solution of $(\beta+t)\lambda = 2^P$ for $\alpha < 0$, and $(\beta+t)\lambda = 2^P(\beta+t)^\alpha\nu(t)$ for $0 \le \alpha \le 1$, respectively. In particular, $(\beta+t_\lambda)\lambda = 2^P$ for $\alpha < 0$ and $(\beta+t_\lambda)\lambda = 2^P(\beta+t_\lambda)^\alpha\nu(t_\lambda)$ for $0 \le \alpha \le 1$ are called separating curves in the quantized time-frequency space, respectively.

It is evident that for $\alpha < 1$, t_λ is essentially bounded since the first eigenvalue of the Dirichlet Laplacian $-\Delta$ is strictly greater than 0. Moreover, the separating curve turns downward as t tends to $+\infty$ in the case of $\alpha < 1$ and turns upward as t tends to $+\infty$ in the case of $\alpha = 1$ when $\nu(t)$ is strictly increasing to $+\infty$. It is worth noticing that the separating curve is in effect a straight line parallel to the time axis in the case of $\alpha = 1$ if $\nu(t)$ is a constant.

Definition

In each zone, we define the corresponding micro-energy matrix in a uniform manner $V(t,\lambda)$. And the precise micro-energies are listed below, $(\mathrm{D}_t = \frac{1}{i}\frac{\partial}{\partial t})$

- In $\mathbb{Z}_1$,

$$V(t,\lambda) := (\lambda\mathcal{F}\{u + \eta F(u)\}, \mathcal{F}\{(1+\eta F'(u))\mathrm{D}_t u\})^T;$$

- In $\mathbb{Z}_2$,

$$V(t,\lambda) := (\lambda A(t)\mathcal{F}\{u + \eta F(u)\}, \mathcal{F}\{(1+\eta F'(u))\mathrm{D}_t u\})^T.$$

Definition

In $\mathbb{Z}_2$, we define the following generalized Schwarz symbol class $S^r\{m_1, m_2\}$. More specifically, for $r \in \mathbb{N}$, we denote $a(t,\lambda) \in S^r\{m_1, m_2\}$

- if for $\forall\ k = 0, \cdots, r$,

$$|a^{(k)}(t,\lambda)| \leq C_k \lambda^{m_1} (\beta + t)^{\alpha} \left(\frac{\nu(t)}{\beta + t} \right)^{m_2 + k}$$

  hold for $\alpha < 0$;

- if for $\forall\ k = 0, \cdots, r$,

$$|a^{(k)}(t,\lambda)| \leq C_k \lambda^{m_1} \left((\beta + t)^{\alpha - 1} \nu(t) \right)^{m_2 + k}$$

  hold for $0 \leq \alpha \leq 1$,

where $m_i \in \mathbb{R}$, $i = 1, 2$. And C_k, $k = 0, \cdots, r$ are positive constants.

We shall denote by OPS^r the class of pseudodifferential operators with symbols in $S^r\{m_1, m_2\}$, endowed with the topology defined by the following norms.

- For $\alpha < 0$,

$$\|a\|_r := \sup_{\substack{0 \le k \le r,\, k \in \mathbb{N} \\ (t, \lambda^2) \in \mathbb{Z}_2}} \left\{ \left| a^{(k)} \lambda^{-m_1} (\beta + t)^{-\alpha} \left(\frac{\nu(t)}{\beta + t} \right)^{-m_2 - k} \right| \right\}.$$

- For $0 \le \alpha \le 1$,

$$\|a\|_r := \sup_{\substack{0 \le k \le r,\, k \in \mathbb{N} \\ (t, \lambda^2) \in \mathbb{Z}_2}} \left\{ \left| a^{(k)} \lambda^{-m_1} \left((\beta + t)^{\alpha - 1} \nu(t) \right)^{-m_2 - k} \right| \right\}.$$

Lemma

As a matter of fact, by using the above definitions, we can immediately check the following rules of symbol calculus, which are crucial for the normal form diagonalization processes.

- $S^{r+1}\{m_1, m_2\} \subset S^r\{m_1, m_2\}$;
- *If* $a \in S^r\{m_1, m_2\}$, $b \in S^r\{k_1, k_2\}$, *then,* $ab \in S^r\{m_1 + k_1, m_2 + k_2\}$;
- *If* $a \in S^r\{m_1, m_2\}$, *then,* $a^{(k)} \in S^{r-k}\{m_1, m_2 + k\}$, *where* $k = 0, \cdots, r$.

- **Step 1:** Micro-energy estimates in $\mathbb{Z}_1$. By keeping in mind the definition of the micro-energy in this zone, we consider the following first order system by applying the iteration method and induction method.

$$\mathrm{D}_t V = \begin{pmatrix} 0 & \lambda \\ \lambda A^2(t) & i \end{pmatrix} V.$$

- **Step 2:** Micro-energy estimates in $\mathbb{Z}_2$. The whole proof is based on three steps of diagonalization. By taking into account the definition of micro-energy in this zone, we study the following first order system by applying the normal form diagonalization procedures for both cases $A(t)\lambda < 0.5$ and $A(t)\lambda > 0.5$.

$$\mathrm{D}_t V = \begin{pmatrix} 0 & A(t)\lambda \\ A(t)\lambda & 0 \end{pmatrix} V + \begin{pmatrix} \frac{A'(t)}{iA(t)} & 0 \\ 0 & i \end{pmatrix} V.$$

First Step of diagonalization: Let

$$\mathscr{M} := \begin{pmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{pmatrix}.$$

Through the invertible transform $V_0 = \mathscr{M} V$, we obtain

$$\mathrm{D}_t V_0 - \overline{\mathscr{D}} V_0 + \overline{\mathscr{K}} V_0$$

$$:= \mathrm{D}_t V_0 - \begin{pmatrix} -A(t)\lambda + 0.5i & -0.5i \\ -0.5i & A(t)\lambda + 0.5i \end{pmatrix} V_0$$

$$- \begin{pmatrix} \frac{A'(t)}{2iA(t)} & \frac{A'(t)}{2iA(t)} \\ \frac{A'(t)}{2iA(t)} & \frac{A'(t)}{2iA(t)} \end{pmatrix} V_0 = 0,$$

where $\overline{\mathscr{K}} \in S^1\{0,1\}$.

Lower frequency part $A(t)\lambda < 0.5$

If there exists an $m \in \mathbb{N}$ such that $A(t)\lambda_k < 0.5$, $k = 1, \cdots, m$, we apply the following diagonalization procedure.

Second Step of diagonalization: Let

$$\Phi := \begin{pmatrix} 0.5i & 0.5i \\ i\sqrt{0.25 - A^2(t)\lambda^2} - A(t)\lambda & -i\sqrt{0.25 - A^2(t)\lambda^2} - A(t)\lambda \end{pmatrix}.$$

By applying the invertible transform $V_0 = \Phi V_2$, in fact, we achieve

$$\mathrm{D}_t V_2 - \Phi^{-1}\mathscr{D}\Phi V_2 + \Phi^{-1}\mathscr{K}\Phi V_2 + \Phi^{-1}\mathrm{D}_t\Phi V_2 = 0.$$

For lower frequencies, it suffices to obtain the final estimates without further diagonalization procedure.

Higher frequency part $A(t)\lambda > 0.5$
Second Step of diagonalization: Let

$$\Phi := \begin{pmatrix} 0.5i & -\dfrac{0.5i}{\lambda} \\ \sqrt{A^2(t)\lambda^2 - 0.25} - A(t)\lambda & \dfrac{\sqrt{A^2(t)\lambda^2 - 0.25} + A(t)\lambda}{\lambda} \end{pmatrix}.$$

By applying the invertible transform $V_0 = \Phi V_2$, we have

$$\mathrm{D}_t V_2 - \Phi^{-1}\overline{\mathscr{D}}\Phi V_2 + \Phi^{-1}\overline{\mathscr{K}}\Phi V_2 + \Phi^{-1}\mathrm{D}_t\Phi V_2 = 0.$$

Third Step of diagonalization: In fact, we aim to construct an invertible matrix $\mathscr{N}_1(t,\lambda) := I + \mathscr{N}^{(1)}(t,\lambda)$ for our purposes. To further the discussion, we introduce the following notations,

$$\mathscr{D} := \Phi^{-1}\overline{\mathscr{D}}\Phi, \quad \mathscr{K} := \Phi^{-1}\overline{\mathscr{K}}\Phi + \Phi^{-1}\mathrm{D}_t\Phi;$$

$$\mathscr{K}^{(0)} := \mathscr{K}, \quad \mathscr{F}^{(0)} := \operatorname{diag}(\mathscr{K}^{(0)});$$

$$\mathscr{N}^{(1)}_{qr} := \mathscr{K}^{(0)}_{qr}/(\tau_q - \tau_r),\ q \neq r,\ \mathscr{N}^{(1)}_{qq} := 0;$$

$$\tau_k := (-1)^k\sqrt{A^2(t)\lambda^2 - 0.25} + 0.5i,\ k = 1, 2;$$

$$\mathscr{K}^{(1)} := (\mathrm{D}_t - \mathscr{D} + \mathscr{K})(I + \mathscr{N}^{(1)}) - (I + \mathscr{N}^{(1)})(\mathrm{D}_t - \mathscr{D} + \mathscr{F}^{(0)}).$$

Let $\mathscr{R}_1 := \mathscr{N}_1^{-1}\mathscr{K}^{(1)}$. Here, $\mathscr{N}^{(1)} \in S^1\{-1,1\}$ and $\mathscr{K}^{(1)} \in S^0\{-1,2\}$. By transforming to $V_1 = \mathscr{N}_1^{-1}V_2$, we consider the following first order system, where $\mathscr{R}_1 \in S^0\{-1,2\}$,

$$(\mathrm{D}_t - \mathscr{D} + \operatorname{diag}(\mathscr{K}) + \mathscr{R}_1)V_1 = 0.$$

Sketch of proof of optimality estimates

Definition

For a sufficiently small $\varepsilon > 0$, we define two functions on $\mathbb{R}$:

$$
\begin{aligned}
w_\varepsilon(t) &:= \sin(10t)\mathrm{e}^{-0.5t}\mathrm{e}^{(1+2\varepsilon)\int_0^t (1-\cos\tau)\sin^2(10\tau)\mathrm{d}\tau}, \\
a_\varepsilon(t) &:= 10.5^2 - (20+40\varepsilon)(1-\cos t)\sin(20t) \\
&\quad -(1+2\varepsilon)\sin t\sin^2(10t) \\
&\quad -(1+4\varepsilon+4\varepsilon^2)(1-\cos t)^2\sin^4(10t).
\end{aligned}
$$

Furthermore, in a period, we have

$$\int_0^{2\pi} (1 - \cos\tau)\sin^2(10\tau)\mathrm{d}\tau = \pi.$$

It is easy to verify the following statement.

Lemma

According to the above definition, $a_\varepsilon, w_\varepsilon \in C^\infty(\mathbb{R})$. Particularly, w_ε is the unique solution of the following Cauchy problem,

$$\begin{cases} w_\varepsilon''(t) + w_\varepsilon'(t) + a_\varepsilon(t)w_\varepsilon(t) = 0, \\ w_\varepsilon(0) = 0, \\ w_\varepsilon'(0) = 10. \end{cases}$$

Definition

We define an oscillating interval as

$$I := [\beta, +\infty) \subset [1, +\infty),$$

and the step for progressive growth of the time

$$\rho := 2^p \pi [\beta],$$

where $[\cdot]$ stands for the integer part. And $p \geq 2$ is a sufficiently large integer such that for any $n \in \mathbb{N}_+$,

$$\frac{1}{2^p n\pi} + 0.5 < \varepsilon < 0.51.$$

It is evident that $\dfrac{\rho}{4\pi} \in \mathbb{N}_+$.

Definition

We define the piece-wise oscillating coefficient $A^2(t)$ as

$$A^2(t) := \begin{cases} 10.5^2, & t \in \ [0, \beta); \\ a_\varepsilon(t-\beta), & t \in [\beta, +\infty). \end{cases}$$

The above definition indicates that, on the one hand, $A^2 \in C^2(\mathbb{R})$ according to the structure of $a_\varepsilon(t)$. On the other hand, let us define

$$d_0 := 55.25 - 98\varepsilon - 16\varepsilon^2,$$

$$d_1 := 145.25 + 98\varepsilon + 16\varepsilon^2.$$

The choice of ε assures that $d_0 > 0.25$. Subsequently, we have

$$0 < d_0 \le \min_{t\in[0,+\infty)} A^2(t) \le \max_{t\in[0,+\infty)} A^2(t) \le d_1 < +\infty.$$

The coefficient A^2 satisfies Assumptions 1-5.

In this section, we consider the 1-D case $\Omega := (0, \pi)$. By applying the energy method for constant coefficients, we reach the following statement for the backward problem.

Lemma

For the Cauchy problem in the interval $[0, \beta]$,

$$\begin{cases} u_{tt} - 10.5^2 u_{xx} + u_t = \dfrac{\eta F''(u)}{1 + \eta F'(u)}(10.5^2 u_x^2 - u_t^2), \\ u(\beta, x) = -\eta F(u(\beta, x)), x = 0, \pi, \\ u_t(\beta, x) = \dfrac{10 \sin x}{1 + \eta F'(u(\beta, x))}, \end{cases}$$

then, the following energy estimates hold, that is,

$$\mathrm{e}^{-2\beta} \mathbf{E}_\eta(u)(0) \leq \mathbf{E}_\eta(u)(\beta) = 50\pi \leq \mathbf{E}_\eta(u)(0).$$

Next, we deal with the forward Cauchy problem in the oscillating interval,

$$\begin{cases} u_{tt} - A^2(t)u_{xx} + u_t = \dfrac{\eta F''(u)}{1+\eta F'(u)}(A^2(t)u_x^2 - u_t^2), \\ u(\beta, x) = -\eta F(u(\beta, x)), x = 0, \pi, \\ u_t(\beta, x) = \dfrac{10 \sin x}{1+\eta F'(u(\beta, x))}, \end{cases}$$

Indeed, we have the explicit representation in the oscillating interval I

$$u(t,x) = G^{-1}\{\sin x w_\varepsilon(t-\beta)\},$$

where G^{-1} is the inverse function of $G(x) = x + \eta F(x)$.

For any $n \in \mathbb{N}_+$, further calculations lead to

$$u(\beta + 0.5n\rho, x) = -\eta F(u(\beta + 0.5n\rho, x)),$$

$$u_t(\beta + 0.5n\rho, x) = \frac{10\mathrm{e}^{0.5n\varepsilon\rho} \sin x}{1 + \eta F'(u(\beta + 0.5n\rho, x))}.$$

For convenience's sake, let β be a positive integer. By recalling the energy definition, we have the corresponding energy estimate,

$$\begin{aligned} \mathbf{E}_\eta(u)(\beta + 0.5n\rho) &= 50\pi\mathrm{e}^{n\varepsilon\rho} \\ &= 50\pi\mathrm{e}^{n2^p\pi\varepsilon\beta} \\ &\geq 50\pi\mathrm{e}^{\beta + 0.5n\rho} \\ &= 50\pi\mathrm{e}^{-\beta}\mathrm{e}^{2\beta + 0.5n\rho}. \end{aligned}$$

It is evident that

$$\mathbf{E}_\eta(u)(\beta + 0.5n\rho) \geq \mathrm{e}^{-3\beta}\mathrm{e}^{2\beta+0.5n\rho}\mathbf{E}_\eta(u)(0).$$

Therefore, once $\mathbf{E}_\eta(u)(0) > 0$, when we choose $0 < c < 1$ and the time sequence

$$\{t_n\}_n = \{\beta + 0.5n\rho\}_n,$$

we conclude the proof of optimality estimates immediately.

The method proposed in this paper is also very efficient in solving certain fractional order equations equipped with suitable boundary conditions, such as the following linear fractional order equation,

$$\begin{cases} u_{tt} + A^2(t)(-\Delta)^\sigma u + \gamma(-\Delta)^\delta u_t = 0, & \text{in } (0,+\infty)\times\Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & \text{on } \Omega, \end{cases}$$

where $\gamma > 0$. For instance, if $\sigma = 1$ and $\delta = 0$, which is the typical wave equation, we use the Dirichlet boundary condition. While if $\sigma = 2$ and $\delta = 0$, which corresponds to the damped plate equation, we apply the boundary condition $u(t,x) = \Delta u(t,x) = 0$ on Γ. If we define the corresponding nonhomogeneous energy as

$$\mathbf{E}(w)(t) := A^2(t)\|(-\Delta)^{\frac{\sigma}{2}} w(t,\cdot)\|_{L^2}^2 + \|w_t(t,\cdot)\|_{L^2}^2,$$

then, we arrive at similar estimates with a modified decay factor $\mathrm{e}^{-\gamma t}$ in the case of $\delta = 0$.

THANKS FOR YOUR ATTENTION