Subelliptic problems with critical Sobolev exponent and Hardy potential on stratified groups

Dedicated to the memory of my dearest professor

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13-th ISAAC Congress
August 2 - August 6, 2021
Ghent, Belgium

We aim to extend to the stratified setting the following results:

Theorem (E. Jannelli, JDE 2000)

Let us consider the problem

$$\begin{cases} -\Delta u - \mu \dfrac{u}{|x|^2} u &= |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega \subset \mathbb{R}^n, \quad 0 \in \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \qquad (P_{\mu,\lambda})$$

The following statements hold:

(a_0) for $\lambda \leq 0$, if Ω is a connected starshaped domain, problem $(P_{\mu,\lambda})$ has no solutions.

(a) If $0 \leq \mu \leq \overline{\mu} - 1$, then $(P_{\mu,\lambda})$ has at least one positive solution for any $0 < \lambda < \lambda_1(\mu)$;

(b) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, there exists $\lambda_* > 0$ such that $(P_{\mu,\lambda})$ has at least one positive solution for $\lambda_* < \lambda < \lambda_1(\mu)$;

(c) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, $\Omega = B(0, R)$, there exists $\lambda_{**} > 0$ such that $(P_{\mu,\lambda})$ has no positive solutions for $0 < \lambda < \lambda_{**}(\mu)$(critical range of μ).

The stratified setting

Def. A **stratified Lie group** $(\mathbb{G}, \circ)$ is a connected simply connected Lie group with stratified Lie algebra, i.e.

$$\mathfrak{g} = \bigoplus_{j=1}^{r} V_j$$

where $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, $[V_1, V_r] = \{0\}$. So V_1 generates by commutation the whole Lie algebra of $\mathbb{G}$.

- A stratified group $(\mathbb{G}, \circ)$ is canonically isomorphic to a homogeneous Lie group (called homogeous Carnot group) which has the form $\mathbb{G} \simeq (\mathbb{R}^N, \circ, \delta_\lambda) = (\mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_r}, \circ, \delta_\lambda)$, with dilations of the form

$$\delta_\lambda(\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(r)}) = (\lambda\, \xi^{(1)}, \lambda^2\, \xi^{(2)}, \cdots, \lambda^r\, \xi^{(r)})$$

- The number

$$Q := \sum_{i=1}^{r} i N_i$$

  is called the *homogeneous dimension* of $\mathbb{G}$.

If $X_j \in \mathfrak{g}$, $j = 1, \dots, N_1$ are the unique vector fields which coincide at 0 with $\frac{\partial}{\partial \xi_j}$, they satisfy $rank(Lie\{X_1, \dots, X_{N_1}\}) = N$ for every $\xi \in \mathbb{R}^N$. The operator

$$\Delta_{\mathbb{G}} = \sum_{j=1}^{N_1} X_j^2$$

is called the canonical sub-Laplacian on $\mathbb{G}$. Any operator

$$\mathcal{L} = \sum_{j=1}^{N_1} Y_j^2$$

where $Y_1, \dots, Y_{N_1}$ is any basis of $span\{Z_1, \dots, Z_{N_1}\}$ is simply called a sub-Laplacian on $\mathbb{G}$. We shall denote $\nabla_{\mathcal{L}} = (Y_1, \dots, Y_{N_1})$
By Hörmander's theorem, a sublaplacian is a hypoelliptic operator.
Moreover, $\mathcal{L}$ is homogeneous of degree 2 with respect of δ_λ.
Let $Q \geq 3$. An $\mathcal{L}$-gauge is a homogeneous norm on $\mathbb{G}$ s.t. $\mathcal{L}(d^{2-Q}) = 0$ in $\mathbb{G} \setminus \{0\}$. If d is an $\mathcal{L}$-gauge, there exists $\beta_d > 0$ s.t. $\Gamma(\xi) = \beta_d d(\xi)^{2-Q}$ is a fundamental solution of $\mathcal{L}$ with pole at 0. (see Folland '75).

Our problem on $\mathbb{G}$

Our aim is to study existence and nonexistence of positive solutions for the problem with Hardy term on bounded domains of $\mathbb{G}$

$$\begin{cases} -\mathcal{L}u - \mu\dfrac{\psi^2}{d^2}u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega \subset \mathbb{G}, \quad 0 \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P^{\mathbb{G}}_{\mu,\lambda})$$

where $\mathcal{L}$ is a sub-Laplacian on a (homogeneous) Carnot group $\mathbb{G}$, d is the $\mathcal{L}$-gauge, $\psi := |\nabla_{\mathcal{L}} d|$ and $0 \leq \mu < \overline{\mu}$, where $\overline{\mu} = \left(\frac{Q-2}{2}\right)^2$ is the best constant in the Hardy inequality on Carnot groups

$$\int_\Omega |\nabla_{\mathcal{L}} u|^2 \, d\xi \geq \overline{\mu} \int_\Omega \psi^2 \frac{|u|^2}{d(\xi)^2} \, d\xi, \qquad \forall u \in C_0^\infty(\Omega)$$

Functional background for subelliptic Hardy's ineq.: wide literature starting from Garofalo-Lanconelli ('90), a complete exposition in the recent monograph by Ruzhansky-Suragan, *Hardy inequalities on homogeneous groups* (2019) and the references therein.

Extending non existence results

We need to implement a Pohozaev-type identity for problem $P^{\mathbb{G}}_{\mu,\lambda}$ on $\mathbb{G}$.

First, notice that the solutions of the problem are singular at the origin, unlike the case $\mu = 0$ [See L., Med.J. Math. 2018]:

Theorem (Local behavior of solutions)

Let Ω be an arbitrary open subset of $\mathbb{G}$, $0 \in \Omega$. If $u \in \mathcal{D}^{1,2}(\Omega)$ is a nonnegative solution to

$$-\mathcal{L}u - \mu\frac{\psi^2}{d^2}u = u^{2^*-1} + \lambda u$$

then

$$u(\xi) \sim \frac{C}{d^{\sqrt{\bar{\mu}}-\sqrt{\bar{\mu}-\mu}}} \qquad \text{as } d(\xi) \to 0.$$

So, the singularity becomes stronger and stronger as $\mu \to \bar{\mu}$.

We obtain the following Pohozaev-type identity

Theorem (L., 2021)

Let $\mathbb{G}$ be a Carnot group and let $\Omega \subset \mathbb{G}$ be a smooth bounded domain, $0 \in \Omega$, and $u \in \mathcal{D}^{1,2}(\Omega) \cap \Gamma^2(\overline{\Omega} \setminus \{0\})$, such that $Zu/d \in L^2(\Omega)$ be a solution of $P^{\mathbb{G}}_{\mu,\lambda}$. Then, the following identity holds

$$\lambda \int_\Omega u^2 \, \mathrm{d}\xi = \frac{1}{2} \int_{\partial\Omega} |\nabla_{\mathcal{L}} u|^2 \langle Z, \nu \rangle \, \mathrm{d}\sigma$$

where Z is the infinitesimal generator of the dilations δ_λ, $\nu = (\nu_1, \ldots, \nu_N)$ is the outward normal to $\partial\Omega$.

The identity is obtained by multiplying the equation for Zu and integrating by parts on approximating domains $\Omega \setminus B_{r_n}(0)$, and then letting $r_n \to 0$. We emphasize that in the Euclidean case the condition Zu/d is implied by the assumption $|\nabla u| \in L^2$, while this is not the case in the stratified setting.

(Recall that $Z = \sum_{i=1}^r \sum_{j=1}^{N_i} i\xi_j^{(i)} \frac{\partial}{\partial \xi_j^{(i)}}$, in the Euclidean case $Z = x \cdot \nabla$)

From the above identity, we get:

Theorem (Nonexistence for $\lambda \leq 0$)

Let $\Omega \subset \mathbb{G}$ be a smooth connected bounded domain, δ_λ-starshaped about the origin (i.e. $\langle Z, \nu \rangle \geq 0$ on $\partial\Omega$). If $\lambda \leq 0$, problem $P^{\mathbb{G}}_{\mu,\lambda}$ has no nonnegative nontrivial solutions $u \in \mathcal{D}^{1,2}(\Omega) \cap \Gamma^2(\overline{\Omega} \setminus \{0\})$, such that $Zu/d \in L^2(\Omega)$.

Remark: The regularity assumptions on the boundary can be somehow weakened.

Extending existence results

A crucial ingredient for the existence result in the Euclidean setting was the knowledge of the ground state solutions to the limit problem on $\mathbb{R}^n$.

$$-\Delta u - \mu\frac{u}{|x|^2} = u^{2^*-1}, \quad u \in H^1(\mathbb{R}^n)$$

It is known (see Terracini, Adv. Differ. Equ. '96) that all positive solutions of the above equation are radial and they have the form

$$U_\varepsilon(x) = \frac{C_\varepsilon}{\left(\varepsilon|x|^{\frac{\gamma'}{\sqrt{\bar\mu}}} + |x|^{\frac{\gamma}{\sqrt{\bar\mu}}}\right)^{\sqrt{\bar\mu}}}, \quad \varepsilon > 0$$

where $\gamma' = \sqrt{\bar\mu} - \sqrt{\bar\mu - \mu}$ and $\gamma = \sqrt{\bar\mu} + \sqrt{\bar\mu - \mu}$.

In the Carnot setting, they are known only for $\mu = 0$ and when G is an Iwasawa-type group (See Jerison-Lee '88, Frank-Lieb 2012, Ivanov-Minchev-Vassilev 2012, Christ-Liu-Zhang 2016).

The limit problem on $\mathbb{G}$

We are led to study qualitative properties of solutions to the limit problem

$$-\mathcal{L}u - \mu\psi^2\frac{u}{d^2} = u^{2^*-1}, \quad u \in \mathcal{D}^{1,2}(\mathbb{G})$$

We prove the following two basic results:

• Ground state solutions do exist, i.e. the best constant

$$S_\mu := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{G}),u\neq 0} \frac{\int_{\mathbb{G}} |\nabla_{\mathcal{L}}u|^2 \, \mathrm{d}\xi - \mu\int_{\mathbb{G}} \frac{\psi^2}{d^2}u^2 \, \mathrm{d}\xi}{\left(\int_{\mathbb{G}} |u|^{2^*} \, \mathrm{d}\xi\right)^{2/2^*}}.$$

is achieved.

• We establish the exact rate of decay at ∞ of ground states, as weak positive solutions to the above equation.

The equivalent weighted problem

Observe that, if $u \in \mathcal{D}^{1,2}(\mathbb{G})$ is a solution of

$$-\mathcal{L}u - \mu\frac{\psi^2}{d^2}u = u^{2^*-1} \ in \ \mathbb{G},$$

then, the function

$$v := d^\alpha u \in \mathcal{D}^{1,2}\left(\mathbb{G}, d^{-2\alpha}\,\mathrm{d}\xi\right), \qquad \alpha = \sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}$$

and satisfies

$$\boxed{-\,\mathrm{div}_{\mathcal{L}}\left(d^{-2\alpha}\nabla_{\mathcal{L}}v\right) = \frac{v^{2^*-1}}{d^{2^*\alpha}} \ in\, \mathbb{G}}$$

Observe that the above equation is the one satisfied by the extremals of the following weighted Sobolev inequality

$$\left(\int_{\mathbb{G}} d^{-2^*\alpha}\,|v|^{2^*}\ d\xi\right)^{\frac{2}{2^*}} \le C_\alpha \int_{\mathbb{G}} d^{-2\alpha}\,|\nabla_{\mathcal{L}}v|^2\ d\xi,\ \ \forall v \in \mathcal{D}^{1,2}\left(\mathbb{G}, d^{-2\alpha}\,\mathrm{d}\xi\right),$$

(The inequality is, in fact, valid for any $\alpha < \sqrt{\bar{\mu}}$)

For the equivalent weighted problem, we obtain the following

Theorem (L., 2021)

Any nonnegative weak solution $v \in \mathcal{D}^{1,2}\left(\mathbb{G}, d^{-2\alpha}\,\mathrm{d}\xi\right)$ to

$$-\mathrm{div}_{\mathcal{L}}\left(d^{-2\alpha}\nabla_{\mathcal{L}} v\right) = \frac{v^{2^*-1}}{d^{2^*\alpha}} \quad \text{in } \mathbb{G} \qquad (P_\alpha)$$

satisfies:

1)

$$v \in L^\infty(\mathbb{G}) \cap L^{\frac{2^*}{2},\infty}(\mathbb{G},\, d^{-2^*\alpha}\,\mathrm{d}\xi).$$

2)

$$v(\xi) \sim \frac{C}{d(\xi)^{Q-2-2\alpha}} \quad \text{for } d \text{ large.}$$

The estimate on the rate of decay requires many technical devices such as reverse Hölder inequalities, Moser-type estimates in weighted Sobolev subelliptic spaces.

Remarks on weak Lebesgue regularity

The sharp weak Lebesgue regularity is a crucial tool in detecting the decay at ∞ of solutions. The idea of such relation goes back to Jannelli-Solimini, Ric. Mat. ('99).

Moreover, as observed by Brasco et alt, Calc. Var. 2015, if u is positive radial and monotone decreasing in $\mathbb{R}^n$ and $u \in L^{p,\infty}(\mathbb{R}^n)$, then

$$u(x) \leq \frac{C}{|x|^{n/p}}, \ |x| \ large$$

. But in the stratified context radial symmetry does not hold for the considered problems, so the weak regularity is only a first step and a refined regularity analysis is needed to obtain the rate of decay of solutions.

From the above asymptotic estimates for the solutions v to the transformed problem, since $u = d^{-\alpha}v$, the desired estimate follows for u, i.e.

Corollary

Let $u \in \mathcal{D}^{1,2}(\mathbb{G})$ be a nonnegative weak solution to the limit problem

$$-\mathcal{L}u - \mu\frac{\psi^2}{d^2}u = u^{2^*-1} \text{ in } \mathbb{G}.$$

Then, u satisfies

$$u(\xi) \sim \frac{1}{d(\xi)^{\sqrt{\bar{\mu}}+\sqrt{\bar{\mu}-\mu}}}, \quad \text{as } d(\xi) \to \infty.$$

Existence results on bounded domains

By means of the asymptotic estimates for entire solutions of the limit problem, we are able to prove the following existence results on bounded domains for our problem

$$\begin{cases} -\mathcal{L}u - \mu \dfrac{\psi^2}{d^2} u = |u|^{2^*-2}u + \lambda u & \text{in } \Omega, \quad 0 \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (P^{\mathbb{G}}_{\mu,\lambda})$$

Theorem (L., 2021)

We can prove that
(A) If $\mu \le \overline{\mu} - 1$, then $(P^{\mathbb{G}}_{\mu,\lambda})$ has at least one positive solution for any $0 < \lambda < \lambda_1(\mu)$;
(B) If $\overline{\mu} - 1 < \mu < \overline{\mu}$, there exists $\lambda_* > 0$ such that $(P^{\mathbb{G}}_{\mu,\lambda})$ has at least one positive solution for $\lambda_* < \lambda < \lambda_1(\mu)$

(C) Is the range $\overline{\mu} - 1 < \mu < \overline{\mu}$ critical? (i.e. is there a class of domains in which the problem has no solutions for λ small?) Still open, work in progress

Idea of the existence proof

Following the Euclidean scheme, a sufficient condition for the existence of a positive solution when $0 < \lambda < \lambda_1(\mu)$ is that

$$S_{\mu,\lambda} := \inf_{u \in \mathcal{D}^{1,2}(\Omega)} \frac{\int_\Omega |\nabla_{\mathcal{L}} u|^2 \, d\xi - \mu \int_\Omega \frac{\psi^2}{d^2} u^2 \, d\xi - \lambda \int_\Omega u^2 \, d\xi}{\left(\int_\Omega |u|^{2^*} \, d\xi\right)^{2/2^*}} < S_\mu, ,$$

since this ensures that $S_{\mu,\lambda}$ is achieved.

This is proved by calculating the ratio in $S_{\mu,\lambda}$ in the one-parameter family

$$u_\epsilon = \phi U_\epsilon,$$

where $\phi \in C_0^\infty(\Omega)$ is a suitable cut-off function, $U > 0$ is a fixed minimizer for S_μ and for $\varepsilon > 0$, U_ε is the abstract family of *concentrating functions*

$$U_\varepsilon(\xi) = \varepsilon^{\frac{2-Q}{2}} U(\delta_{\frac{1}{\varepsilon}}(\xi)).$$

By only using the decay of U_ε, we are able to obtain the estimates, as $\varepsilon \to 0$:

$$\int_\Omega \left(|\nabla_{\mathcal{L}} u_\varepsilon|^2 - \mu \frac{\psi^2}{d^2} u_\varepsilon^2 \right) \mathrm{d}\xi = S_\mu^{\frac{Q}{2}} + O(\varepsilon^{Q-2})$$

$$\int_\Omega u_\varepsilon^{2^*} \mathrm{d}\xi = S_\mu^{\frac{Q}{2}} + O(\varepsilon^{Q})$$

$$\int_\Omega u_\varepsilon^2 \, \mathrm{d}\xi = \begin{cases} C\,\varepsilon^2 + O(\varepsilon^{2\sqrt{\overline{\mu}-\mu}}) & \text{if } 0 \le \mu < \overline{\mu} - 1 \\ C\,\varepsilon^2 |\log \varepsilon| + O(\varepsilon^2) & \text{if } \mu = \overline{\mu} - 1. \end{cases}$$

By using the above expansions, we can conclude.

The existence result for λ in a left neighborhood of $\lambda_1(\mu)$ and $\overline{\mu} - 1 < \mu < \overline{\mu}$ is achieved by a bifurcation argument.

Analogous problem for Grushin-type operators

We just recall that we have performed an analogous qualitative analysis of solutions to the problem

$$\begin{cases} -(\Delta_x u + |x|^{2\alpha}\Delta_y u) - \mu \dfrac{\psi^2}{d^2} u &= |u|^{2^*-2}u \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad \text{on } \partial\Omega \end{cases}$$

where $z = (x, y) \in \Omega \subset \mathbb{R}^m \times \mathbb{R}^n$ arbitrary open set, $0 \in \Omega$, d is the Grushin gauge defined by

$$d(z) = (|x|^{2(\alpha+1)} + (\alpha+1)^2|y|^2)^{\frac{1}{2(\alpha+1)}},$$

$\psi := |\nabla_\alpha d|$, $2^* = \frac{2Q}{Q-2}$, $Q = m + (\alpha+1)n$

$0 \leq \mu < \overline{\mu}$, where $\overline{\mu} = \left(\frac{Q-2}{2}\right)^2$ is the best constant in the Hardy inequality for the Grushin gradient

$$\int_\Omega |\nabla_\alpha u|^2 \,\mathrm{d}z \geq \overline{\mu} \int_\Omega \psi^2 \frac{|u|^2}{d(z)^2} \,\mathrm{d}z, \qquad \forall u \in C_0^\infty(\Omega).$$

The Grushin case was easier to treat than the stratified case, since we could use the related Kelvin-type transform

$$u^*(z) = d(z)^{2-Q} u(\sigma(z)), \quad \text{where } \sigma(z) = \delta_{d(z)^{-2}}(z), \quad z \neq 0.$$

which allows to obtain the behavior at ∞ directly from the behavior at 0, as in the Euclidean case [see L., Ann. Mat. Pura App. 2019]:

Theorem

Let $\Omega \subset \mathbb{R}^N$ be a neighborhood of ∞, $0 \in \Omega$. If $u \in \mathcal{D}^{1,2}_\alpha(\Omega)$ is a positive solution to

$$-\left(\Delta_x u + |x|^{2\alpha} \Delta_y u\right) - \mu \frac{\psi^2}{d^2} u = u^{2^*-1} \ in \ \Omega,$$

then u satisfies

$$u(z) \sim \frac{1}{d(z)^{\sqrt{\bar{\mu}} - \sqrt{\bar{\mu} - \mu}}}, \quad as \ d(z) \to 0,$$
$$u(z) \sim \frac{1}{d(z)^{\sqrt{\bar{\mu}} + \sqrt{\bar{\mu} - \mu}}}, \quad as \ d(z) \to \infty.$$

Some future developments and work in progress:

- 1) to study the quasilinear version of the problem

$$\begin{cases} -\mathcal{L}_p u - \mu \frac{\psi^p}{d^p}|u|^{p-1}u = |u|^{p^*-2}u + \lambda |u|^{p-2}u & \text{in } \Omega \subset \mathbb{G} \text{ bounded} \\ u \in \mathcal{D}^{1,p}(\Omega) \end{cases}$$

  which involves the study of the associated limit problem

$$-\mathcal{L}_p u - \mu \frac{\psi^p}{d^p}|u|^{p-1}u = |u|^{p^*-2}u \text{ in } \mathbb{G}$$

  [The case $\mu = 0$ was studied in L., JMAA 2019]

- 2) to extend the study of the qualitative properties of weak solutions to more general weighted problems of Caffarelli-Kohn-Nirenberg type:

$$-\text{div}_{\mathcal{L}}\left(d^{-2a}\nabla_{\mathcal{L}} u\right) = d^{-bp}|u|^{p-2}u \text{ in } \mathbb{G}$$

  for all admissible a, b, p.

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Thank you for your attention