# Recent results on semilinear wave equations with space or time dependent damping

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# Problem Under Consideration

Consider the following Cauchy problem of wave equation with time or space dependent damping

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^{\alpha}(1+|x|)^{\beta}} u_t = |u|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{0.1}$$

where $\mu > 0, \alpha, \beta \in \mathbb{R}$ are constants and $n \geq 2$, the initial data $f(x), g(x)$ are compactly supported.

# Why these problems?

System (0.1) can be used to model the wave travel in a nonhomogeneous gas with damping, and the time or space dependent coefficients imply that the friction may vary with time or position, see Ikawa(2000, Monographs) and Ikehata-Todorova-Yordanov(2009).

The equation admits both “wave"(hyperbolic) and “heat" (parabolic) phenomenon, people want to figure out the exact asymptotic behavior.

If we consider the compressible Euler equation with space or time dependent damping, a similar equation for the density can be obtained, so it is possible to apply the method for this problem.

# Linear Case

$$
\begin{cases}
u_{tt} - \Delta u + \dfrac{\mu}{(1+|x|)^\beta} u_t = 0, & \text{in } [0,T) \times \mathbb{R}^n, \\
u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in \mathbb{R}^n.
\end{cases} \tag{0.2}
$$

Based on the known results, we may classify the linear problem (0.2) into three cases, due to the value of decay rate $\beta$

| | | |
|---|---|---|
| $\beta \in (-\infty, 1)$ | effective | solution behaves like that of heat equation |
| $\beta = 1$ | scaling invariant weak damping | the asymptotic behavior depends on $\mu$ |
| $\beta \in (1, \infty)$ | scattering | solution behaves like that of wave equation without damping |

Ikehata, Takeda, Todorova, Yordanov, Mochizuki, Radu, Wakasugi,$\cdots$.

# Different From Time Dependent Case

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^{\beta}} u_t = 0, & \text{in } [0,T) \times \mathbb{R}^n, \\ u(x,0) = f(x), \quad u_t(x,0) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{0.3}$$

| | | |
|---|---|---|
| $\beta \in (-\infty, -1)$ | overdamping | solution does not decay to zero |
| $\beta \in [-1, 1)$ | effective | solution behaves like that of heat equation |
| $\beta = 1$ | scaling invariant weak damping | the asymptotic behavior depends on $\mu$ |
| $\beta \in (1, \infty)$ | scattering | solution behaves like that of wave equation without damping |

Hosono, Ogawa, Marcati, Nishihara, Wirth(04-08), $\cdots$.

# Two Critical Powers for Small Data Cauchy Problem

Critical power $p_c(n)$: if $p > p_c(n)$, all small data solutions are global; while if $1 < p \leq p_c(n)$, small data(positive) solutions will blow up in a finite time.

- Strauss exponent $p_S(n) \Rightarrow u_{tt} - \Delta u = |u|^p$, the positive root of the quadratic equation

$$\gamma(p, n) = 2 + (n+1)p - (n-1)p^2 = 0.$$

- Fujita exponent $p_F(n) = 1 + \frac{2}{n} \Rightarrow u_t - \Delta u + u_{tt} = |u|^p$.

# "Strauss"(wave) or "Fujita"(heat)

For

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^{\alpha}(1+|x|)^{\beta}} u_t = |u|^p, & (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(x,0) = \varepsilon f(x), \quad u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^n, \end{cases} \tag{0.4}$$

if the critical power equals to or at least is related to $p_S(n)$, we say the equation admits Strauss or “wave" exponent, while if it has only connection to $p_F(n)$, we then say the problem admits Fujita or “heat" exponent.

# Known Results

The study of such problem has a boardline 2015: before this year, people focus on the “heat" phenomenon, while after that “wave" phenomenon attracts more and more attention.

- $\frac{\mu}{(1+t)^\alpha} u_t (\beta = 0)$ and $\alpha \in [-1, 1)$, the critical power is $p_F(n) = 1 + \frac{2}{n}$, see D'Abbicco-Lucente-Reissig(2013), Li-Zhou(1995), Lin-Nishihara-Zhai(2012), Todorova-Yordanov(2001), Zhang(2001), $\cdots$. Sharp lifespan estimate for $\alpha = 0$: Li-Zhou(1995), Nishihara(2003), Ikeda-Ogawa(2016), Lai-Zhou(2019, JMPA).
- $\frac{\mu}{(1+t)^\alpha} u_t (\beta = 0)$, $\alpha = 1$ and $\mu$ is large, thus $\mu \geq 5/3 (n = 1), 3 (n = 2), n + 2 (n \geq 3)$, the critical power is still $p_F(n) = 1 + \frac{2}{n}$, see D'Abbicco(2015), D'Abbicco-Lucente(2013), Wakasugi(2014), $\cdots$.

# Known Results

- $\frac{\mu}{(1+t)^{\alpha}}u_t(\beta=0)$, $\alpha=1$ and $\mu=2$, the critical power is $p_S(n+2)$, see D'Abbicco-Lucente(2015), D'Abbicco-Lucente-Reissig(2015), Kato-Sakuraba(2019), Lai(2020, Advanced Studies in Pure Mathematics), Palmieri(2019), Wakasa(2016), $\cdots$.

  Fujita$\rightarrowtail$ Strauss, what will happen if $\mu$ becomes smaller?
- $\frac{\mu}{(1+t)^{\alpha}}u_t(\beta=0)$, $\alpha=1$ and $0<\mu<\frac{n^2+n+2}{2(n+2)}$, the solution blows up for $1<p<p_S(n+2\mu)$, see Lai-Takamura-Wakasa(2017, JDE), which was improved(to $1<p<p_S(n+\mu), 0<\mu<\frac{n^2+n+2}{n+2}$) by Ikeda-Sobajima(2018), Tu-Lin(2019), Palmieri-Reissig(2019 Lai-Schiavone-Takamura(2020, JDE), $\cdots$,.
- $\frac{\mu}{(1+t)^{\alpha}}u_t(\beta=0)$, $\alpha=1$ and $\frac{3}{2}\le\mu<2$, there exists global radial solution in $3-D$, see Lai-Zhou(2021, NA).
- $\frac{\mu}{(1+t)^{\alpha}}u_t(\beta=0)$, $\alpha=1$, what about the "middle value" $\mu$?

# Known Results

- $\frac{\mu}{(1+t)^\alpha}u_t(\beta = 0)$, $\alpha > 1$, we are “almost" sure that the critical power is $p_S(n)$, see Lai-Takamura(2018, Nonlinear Anal.), Wakasa-Yordanov(2019), Liu-Wang(2020), $\cdots$.
- $\frac{\mu}{(1+t)^\alpha}u_t(\beta = 0)$, $\alpha < -1$, there are global solutions for all $p > 1$, see Ikeda-Wakasugi(2020).
- $\frac{\mu}{(1+|x|)^\beta}u_t(\alpha = 0)$, $\beta < 1$, the critical power is $p_c(n) = 1 + \frac{2}{n-\beta}$, see Ikehata-Todorova-Yordanov(2009), Nishihara(2010), Nishihara-Sobajima-Wakasugi(2018), $\cdots$.
- $\frac{\mu}{(1+|x|)^\beta}u_t(\alpha = 0)$, $\beta = 1$ and $\mu$ is large, the critical power is $p_c(n) = 1 + \frac{2}{n-1}$(Fujita type), see Li(2013) for $\mu \geq n$.
- $\frac{\mu}{(1+|x|)^\beta}u_t(\alpha = 0)$, $\beta = 1$ and $\mu$ is small, the critical power will move to (Strauss type), see the blow-up result by Ikeda-Sobajima(2020)($\frac{\mu}{|x|}u_t$) for $n \geq 3, 0 \leq \mu < \frac{(n-1)^2}{n+1}$ and $\frac{n}{n-1} < p \leq p_S(n+\mu)$.

# Known Results

- Georgiev-Kubo-Wakasa(2019) showed that the critical power for the radial solutions is the shifted Strauss exponent $p = p_S(3+2)$ in $\mathbb{R}^3$, with damping($D(r)$) and potential($v(r)$) coefficients satisfying: $D(r)$ is a positive decreasing function in $C([0,\infty)) \cap C^1(0,\infty)$ and $D(r) = 2/|x|$ for $r \geq r_0 > 0$, $V(r) = -D'(r)/2 + D^2(r)/4$.
- Dai-Kubo-Sobajima(2021) obtained the upper bound of the lifespan with scale-invariant "critical" damping and potential $D(x) = \frac{d_\infty}{|x|}$, $V(x) = \frac{v_\infty}{|x|^2}$ for $0 \leq d_\infty < n - 1 + 2\rho(v_\infty)$, $v_\infty > -(n-2)^2/4$ and

$$\frac{n+\rho(v_\infty)}{n+\rho(v_\infty)-1} < p \leq p_c = \max(p_S(n+d_\infty), p_G(n+\rho(v_\infty))),$$

where $\rho(v_\infty) := \sqrt{\left(\frac{n-2}{2}\right)^2 + v_\infty} - \frac{n-2}{2}$.

# Known Results

What will happen if $\beta > 1$ and $\alpha = 0$?

- In Ikehata-Todorova-Yordanov(2009):

  as $t \to \infty$ if $V(x) = O(|x|^{-1-\delta})$ with $\delta > 0$. In this case we expect that equation (1.1) loses its "parabolicity" asymptotic effects and turns back to the regime of pure wave equation. Respectively, we expect that the critical exponent $p_c(N,\alpha)$ of the damped wave equation in

  the case of fast decaying potential $\alpha > 1$ jumps to the critical exponent of the wave equation–Strauss' number $p_w(N)$. Namely, $p_c(N,\alpha) = p_w(N)$ for any $\alpha > 1$. The proof of both parts –

- In Nishihara-Sobajima-Wakasugi(2018):

  **Conjecture**

  (i) For $\alpha < \min\{2, N\}$, $\beta > -1$ with $\alpha + \beta < 1$, the critical exponent is $p_c = 1 + \frac{2}{N-\alpha}$.

  (ii) For $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta = 1$, the equation has scale-invariance and the critical exponent will depend on $a_0$.

  (iii) For $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 1$, the critical exponent is given by the Strauss number $p_c = p_S(N)$.

# Known Results

- The only known result for this case is due to Metcalfe-Wang(2017), in which they obtained global existence result for $p > p_S(n)(n = 3, 4)$ if $\beta > 1$ and $\mu$ is small enough.

# Main Results for $\beta > 2$

$\frac{\mu}{(1+|x|)^\beta} u_t (\alpha = 0)$ and $\beta > 2$.

## Theorem (Lai-Tu(2020 JMAA))

*Let $\beta > 2$ and $1 < p < p_S(n), n \geq 3$. Assume that both $f \in H^1(\mathbb{R}^n)$ and $g \in L^2(\mathbb{R}^n)$ are non-negative and do not vanish identically. Then there exists a constant $\varepsilon_0 = \varepsilon_0(f, g, n, p, \mu, \beta) > 0$ such that $T$ has to satisfy*

$$T \leq \begin{cases} C\varepsilon^{-\frac{2(p-1)}{n+1-(n-1)p}} & \text{for } 1 < p \leq \frac{n}{n-1}, \\ C\varepsilon^{-2p(p-1)/\gamma(p,n)} & \text{for } \frac{n}{n-1} < p < p_S(n) \end{cases} \tag{0.5}$$

*for $0 < \varepsilon \leq \varepsilon_0$.*
*If $p = p_S(n)$, then*

$$T \leq \exp\left(C\varepsilon^{-p(p-1)}\right) \tag{0.6}$$

*for $0 < \varepsilon \leq \varepsilon_0$.*

# Energy Solution

we say that $u$ is an energy solution of (0.1) on $[0, T)$ if

$$u \in C([0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T), L^2(\mathbb{R}^n)) \cap L^p_{\mathrm{loc}}(\mathbb{R}^n \times (0, T))$$

satisfies $u(0, x) = \varepsilon f(x), u_t(0, x) = \varepsilon g(x)$ and

$$\begin{aligned}
&\varepsilon \int_{\mathbb{R}^n} g(x)\Psi(0, x)dx + \varepsilon \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} f(x)\Psi(0, x)dx \\
&+ \int_0^T \int_{\mathbb{R}^n} |u|^p \Psi(t, x)dxdt \\
= &- \int_0^T \int_{\mathbb{R}^n} u_t(t, x)\Psi_t(t, x)dxdt + \int_0^T \int_{\mathbb{R}^n} \nabla u(t, x) \cdot \nabla \Psi(t, x)dxdt \\
&- \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} u(t, x)\Psi_t(t, x)dxdt
\end{aligned} \tag{0.7}$$

for any $\Psi(t, x) \in C_0^\infty([0, T) \times \mathbb{R}^n)$.

# Main Idea of Proof

A key observation:
if we have $\phi(x)$ such that

$$\Delta\phi - \frac{\mu}{(1+|x|)^{\beta}}\phi = \phi, \tag{0.8}$$

which was first introduced by Yordanov-Zhang(2005), then $\Phi(t,x) = e^{-t}\phi(x)$ solves

$$\partial_t^2\Phi - \Delta\Phi - \frac{\mu}{(1+|x|)^{\beta}}\partial_t\Phi = 0.$$

# Proof of (0.5)

**Case 1.** $\frac{n}{n-1} < p < p_S(n)$

- Choose the test function $\Psi = \eta_T^{2p'}(t)$ with

$$\eta(t) = \begin{cases} 1 & \text{for } t \leq \frac{1}{2}, \\ \textit{decreasing} & \text{for } \frac{1}{2} < t < 1, \\ 0 & \text{for } t \geq 1 \end{cases}$$

and

$$\eta_T(t) = \eta\left(\frac{t}{T}\right), \quad T \in (1, T_\varepsilon).$$

to get

$$\varepsilon \int_{\mathbb{R}^n} g(x)dx + \varepsilon \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} f(x)dx + \int_0^T \int_{\mathbb{R}^n} |u|^p \eta_T^{2p'} dxdt$$
$$= \int_0^T \int_{\mathbb{R}^n} u\partial_t^2 \eta_T^{2p'} dxdt - \int_0^T \int_{\mathbb{R}^n} \frac{\mu}{(1+|x|)^\beta} u\partial_t \eta_T^{2p'} dxdt$$

# Proof of (0.5)

and furthermore

$$C\varepsilon + \int_0^T \int_{\mathbb{R}^n} |u|^p \eta_T^{2p'} dxdt \le CT^{n-1-\frac{2}{p-1}}. \tag{0.9}$$

- Two Lemmas

**Lemma**

*If $\beta > 0$, then for any $\alpha \in \mathbb{R}$ and a fixed constant $R$, we have*

$$\int_0^{t+R} (1+r)^\alpha e^{-\beta(t-r)} dr \le C(t+R)^\alpha. \tag{0.10}$$

# **Proof of** (0.5)

**Lemma**

**(Lemma 3.1 in Yordanov and Zhang(2005))**. *Assuming that* $\beta > 2$*, then the following equation*

$$\Delta\phi(x) - \frac{\mu}{(1+|x|)^\beta}\phi(x) = \phi(x), \quad x \in \mathbb{R}^n \tag{0.11}$$

*admits a solution satisfying*

$$0 < \phi(x) \leq C(1+|x|)^{-\frac{n-1}{2}} e^{|x|}. \tag{0.12}$$

# Proof of (0.5)

- Choose the test function $\Psi(t,x) = \eta_T^{2p'}(t)e^{-t}\phi(x)$ to get

$$
\begin{aligned}
&\varepsilon \int_{\mathbb{R}^n} g(x)\phi(x)dx + \varepsilon \int_{\mathbb{R}^n} \left(1 + \frac{\mu}{(1+|x|)^\beta}\right) f(x)\phi(x)dx \\
&+ \int_0^T \int_{\mathbb{R}^n} |u|^p \eta_T^{2p'} \Phi dxdt \\
= &\int_0^T \int_{\mathbb{R}^n} u \left(\partial_t^2 \eta_T^{2p'} \Phi + 2\partial_t \eta_T^{2p'} \partial_t \Phi - \frac{\mu}{(1+|x|)^\beta} \partial_t \eta_T^{2p'} \Phi\right) dxdt
\end{aligned}
$$

and furthermore

$$
C\varepsilon \le C T^{-1+(n-\frac{n-1}{2}p')\frac{1}{p'}} \left(\int_0^T \int_{\mathbb{R}^n} \eta_T^{2p'} |u|^p dxdt\right)^{\frac{1}{p}},
$$

which is actually

$$
(C\varepsilon)^p \, T^{n-\frac{n-1}{2}p} \le \int_0^T \int_{\mathbb{R}^n} \eta_T^{2p'} |u|^p dxdt. \tag{0.13}
$$

# Proof of (0.5)

**Case 2.** $1 < p \leq \frac{n}{n-1}$

- Choose the test function $\Psi(t,x) = \eta_T^{2p'}(t)e^{-t}\phi(x)$ to get

$$\varepsilon \int_{\mathbb{R}^n} g(x)\phi(x)dx + \varepsilon \int_{\mathbb{R}^n} \left(1 + \frac{\mu}{(1+|x|)^\beta}\right) f(x)\phi(x)dx$$
$$+ \int_0^T \int_{\mathbb{R}^n} |u|^p \eta_T^{2p'} \Phi dxdt$$
$$\leq CT^{-1+\frac{n+1}{2p'}} \left(\int_0^T \int_{\mathbb{R}^n} \eta_T^{2p'} \Phi |u|^p dxdt\right)^{\frac{1}{p}}$$
$$\leq CT^{-p'+\frac{n+1}{2}} + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \eta_T^{2p'} \Phi |u|^p dxdt,$$

this yields

$$T \leq C\varepsilon^{-\frac{2(p-1)}{n+1-(n-1)p}}, \quad \textit{for } 1 < p < \frac{n+1}{n-1}. \tag{0.14}$$

# Proof of (0.6)

- **Lemma**

  *Assuming*

  $$V(x) = \frac{\mu}{(1+|x|)^\beta}$$

  *and* $\beta > 2$. *Then for given* $\eta \in [0,1]$, *there exists function* $\psi_\eta \in C^2(\mathbb{R}^n)$ *satisfying*

  $$\Delta\psi_\eta - \eta V \psi_\eta = \eta^2 \psi_\eta \tag{0.15}$$

  *such that*

  $$\psi_\eta(x) \sim \varphi_\eta(x) := \int_{\mathbb{S}^{n-1}} e^{\eta x \omega} d\omega (\sim |\eta x|^{\frac{1-n}{2}} e^{|\eta x|}). \tag{0.16}$$

The proof of the above lemma is parallel to that of Lemma 3.1 in Yordanov and Zhang(2005).

# Proof of (0.6)

- Choose the test function

$$b_q(t,x) = \int_0^1 e^{-\eta t}\psi_\eta(x)\eta^{q-1}d\eta, \quad q > 0$$

with

**Lemma**

*(i)* $b_q(t,x)$ *satisfies following identities*

$$\frac{\partial}{\partial t}b_q(t,x) = -b_{q+1}(t,x), \;\; \frac{\partial^2}{\partial t^2}b_q(t,x) = b_{q+2}(t,x)$$

$$\Delta b_q(t,x) = V \cdot b_{q+1}(t,x) + b_{q+2}(t,x),$$

$$\partial_t^2 b_q - \Delta b_q - V\partial_t b_q = 0.$$

*(ii)*

$$b_q(t,x) \sim \begin{cases} (t+R+|x|)^{-q} \text{ for } 0 < q < \frac{n-1}{2}, \\ (t+R+|x|)^{\frac{-n+1}{2}}(t+R-|x|)^{\frac{n-1}{2}-q} \text{ for } q > \frac{n-1}{2}. \end{cases}$$

# Proof of (0.6)

- Introduce

$$\theta(t) = \begin{cases} 0 & \text{for } t < \frac{1}{2}, \\ \eta(t) & \text{for } t \geq \frac{1}{2}, \end{cases} \qquad \theta_M(t) := \theta(\frac{t}{M}).$$

For $M \in (1, T)$, as above one can get

$$\int_0^T \int_{\mathbb{R}^n} \theta_M^{2p'} |u|^p dxdt \geq \int_0^M \int_{\mathbb{R}^n} \theta_M^{2p'} |u|^p dxdt \geq C\varepsilon^p M^{n-\frac{n-1}{2}p}. \tag{0.17}$$

# Proof of (0.6)

- Set

$$Y[w](M) = \int_1^M \left( \int_0^T \int_{\mathbb{R}^n} w(t,x) \theta_\sigma^{2p'}(t) dxdt \right) \sigma^{-1} d\sigma,$$

then for $q = \frac{n-1}{2} - \frac{1}{p}$ we have

$$\begin{aligned} M \frac{d}{dM} Y\left[|u|^p b_q(t,x)\right](M) &= \int_0^T \int_{\mathbb{R}^n} \theta_M^{2p'} b_q |u|^p dxdt \\ &\geq C M^{-\left(\frac{n-1}{2} - \frac{1}{p}\right)} \int_0^T \int_{\mathbb{R}^n} \theta_M^{2p'} |u|^p dxdt \\ &\geq C \varepsilon^p, \end{aligned} \tag{0.18}$$

where we used the fact

$$n - \frac{n-1}{2} p = \frac{n-1}{2} - \frac{1}{p}$$

for $p = p_S(n)$.

# Proof of (0.6)

- A key inequality

$$
\begin{aligned}
Y^p(M) \leq & C \int_0^T \int_{\mathbb{R}^n} \eta_M^{2p'} b_q |u|^p dxdt \\
= & C \int_0^T \int_{\mathbb{R}^n} \left( \partial_t^2 u - \Delta u + \frac{\mu}{(1+|x|)^\beta} \partial_t u \right) b_q \eta_M^{2p'} \\
\leq & C \int_0^T \int_{\mathbb{R}^n} u \left( 2\partial_t b_q \partial_t \eta_M^{2p'} + b_q \partial_t^2 \eta_M^{2p'} - \frac{\mu}{(1+|x|)^\beta} b_q \partial_t \eta_M^{2p'} \right) \\
\leq & CM (\log M)^{p-1} \, Y'(M).
\end{aligned}
\tag{0.19}
$$

# Proof of (0.6)

- Using the following lemma with $p_1 = p_2 = p$ and $\delta = \varepsilon^p$

**Lemma**

**(Lemma 3.10 in Ikeda-Sobajima-Wakasa(2019))**. *Let* $2 < t_0 < T$. $0 \leq \phi \in C^1([t_0, T))$. *Assume that*

$$\begin{cases} \delta \leq K_1 t\phi'(t), \quad t \in (t_0, T), \\ \phi(t)^{p_1} \leq K_2 t(\log t)^{p_2-1}\phi'(t), \quad t \in (t_0, T) \end{cases}$$

*with* $\delta, K_1, K_2 > 0$ *and* $p_1, p_2 > 1$. *If* $p_2 < p_1 + 1$, *then there exists positive constants* $\delta_0$ *and* $K_3$ *(independent of* $\delta$*) such that*

$$T \leq \exp\left(K_3 \delta^{-\frac{p_1-1}{p_1-p_2+1}}\right)$$

*when* $0 < \delta < \delta_0$.

# Main Results for $\beta > 1$

**Theorem (Lai-Liu-Tu-Wang, arXiv:2102.10257)**

*The above result can be generalized to* $\beta > 1$ *and* $\mathbb{R}^n (n \geq 2)$.

# The Key Point for the Proof

**Lemma** (Lai-Liu-Tu-Wang, arXiv:2102.10257)

*Let $n \geq 2$, $\beta > 1$, $\mu \geq 0$. Suppose $D(x) = \frac{\mu}{(1+|x|)^\beta}$. Then there exists $c_1 \in (0,1)$ such that for any $0 < \lambda \leq 1$, there is a $C^2$ solution of*

$$\Delta\phi_\lambda - \lambda D(x)\phi_\lambda = \lambda^2\phi_\lambda \tag{0.20}$$

*satisfying*

$$c_1\langle\lambda|x|\rangle^{-\frac{n-1}{2}}e^{\lambda|x|} < \phi_\lambda(x) < c_1^{-1}\langle\lambda|x|\rangle^{-\frac{n-1}{2}}e^{\lambda|x|}. \tag{0.21}$$

**Remark**

*Actually, the above lemma holds for $D(x) = D(|x|) \in C(\mathbb{R}^n) \cap C^\alpha(B_\delta)$ for some $\alpha$, $\delta > 0$ and $0 \leq D(x) \leq \frac{\mu}{(1+|x|)^\beta}$.*

# Proof of the Key Lemma

We finish the proof by dividing $\mathbb{R}^n$ into two parts: $B_{1/\lambda}$ and $\mathbb{R}^n \setminus B_{1/\lambda}$, and setting $\phi_\lambda(1/\lambda) = 1$.

**Case 1. Inside the ball $B_{1/\lambda}$**

Consider the Dirichlet problem within $B_{1/\lambda}$

$$\begin{cases} \Delta\phi_\lambda - \lambda D(x)\phi_\lambda = \lambda^2\phi_\lambda, x \in B_{1/\lambda}, \\ \phi_\lambda|_{\partial B_{1/\lambda}} = 1. \end{cases} \tag{0.22}$$

Prove uniform lower bound for $\phi_\lambda$ in $B_{1/\lambda}$ by two main steps.

- Upper bound estimates for $\partial_r f_\lambda = \partial_r \phi_\lambda(x/\lambda)$, which can be done directly by using the equation.
- Convergence of $f_\lambda$, which will be done by using Arzela-Ascoli theorem.

# Proof of the Key Lemma

## Case 2. Outside the ball $B_{1/\lambda}$

Let $\phi_\lambda = r^{-\frac{n-1}{2}} y$, then $y$ satisfies

$$\begin{cases} y'' - \lambda^2 y - \Big(\frac{(n-1)(n-3)}{4r^2} + \lambda D(r)\Big) y = 0, \\ y(\frac{1}{\lambda}) = \lambda^{-\frac{n-1}{2}}, \\ y'(\frac{1}{\lambda}) = \frac{n-1}{2}\lambda^{-\frac{n-3}{2}} + \lambda^{-\frac{n-1}{2}}\phi'(\frac{1}{\lambda}) \in \big(0, C_2\lambda y(\frac{1}{\lambda})\big). \end{cases} \tag{0.23}$$

Then using the lemma by Liu-Wang(stated below) with $K = 1, \epsilon = 1, \lambda_0 = 1$ we may get

$$y \backsimeq \lambda^{-\frac{n-1}{2}} e^{\lambda r}, r\lambda \geq 1,$$

and hence

$$\phi_\lambda \backsimeq (\lambda|x|)^{-\frac{n-1}{2}} e^{\lambda|x|}, \lambda|x| \geq 1.$$

# Proof of the key Lemma

Lemma (Lemma 3.1 in Liu-Wang(2019))

*Let* $\lambda \in (0, \lambda_0]$, $\delta_0 \in (0, 1)$, $\varepsilon > 0$, $y_0 > 0$, $K \in (\delta_0, \delta_0^{-1})$,

$$\|K'\|_{L^1([\varepsilon\lambda_0^{-1},\infty))} \leq \delta_0^{-1}, \|G\|_{L^1([\varepsilon\lambda^{-1},\infty))} \leq \delta_0^{-1}\lambda, \forall\lambda \in (0, \lambda_0].$$

$$\begin{cases} y'' - \lambda^2 K^2(r)y + G(r)y = 0, r > \varepsilon\lambda^{-1}, \\ y(\varepsilon\lambda^{-1}) = y_0, y'(\varepsilon\lambda^{-1}) = y_1 \in (0, \delta_0^{-1}\lambda y_0), \end{cases} \tag{0.24}$$

*Then for any solution* $y$ *with* $y, y' > 0$, *we have the following uniform estimates, independent of* $\lambda \in (0, \lambda_0]$,

$$y \simeq y_0 e^{\lambda \int_{\varepsilon/\lambda}^r K(\tau)d\tau}, \ r \geq \varepsilon\lambda^{-1}.$$

*Assume in addition* $1 - \lambda^{-2}K^{-2}G \in (\delta_0, \delta_0^{-1})$, *then the solution* $y$ *to* (0.24) *satisfies* $y, y' > 0$ *and we have*

$$y' \simeq y_1 + y_0\lambda(e^{\lambda \int_{\varepsilon/\lambda}^r K(\tau)d\tau} - 1).$$

# Global Existence Result for $\alpha = 1, \beta = 0$ in 3-D

**Theorem (Lai-Zhou(2021 Nonlinear Analysis))**

*Consider the Cauchy problem* (0.1) *with $\alpha = 1, \beta = 0$ in 3-D. There exists global radial solution for $p > p_S(3 + \mu)$ and $3/2 \leq \mu < 2$.*

**Remark**

*In some sense, by combining the known blow-up results for $\alpha = 1, \beta = 0$, we may confirm that $p_S(3 + \mu)$ is indeed the critical power at least for $3/2 \leq \mu \leq 2$ in 3-D.*

# Key Steps of Proof

- By appropriate transformation, the original equation(radial) can be rewritten as

$$\phi_{u\bar{u}} + \frac{\mu(2-\mu)\phi}{4(u+\bar{u})^2} = \frac{|\phi|^p}{(u-\bar{u})^{p-1}(u+\bar{u})^{\frac{\mu(p-1)}{2}}} \triangleq G(u,\bar{u}), \quad (0.25)$$

  where

$$u = \frac{t+2+r}{2}, \overline{u} = \frac{t+2-r}{2}.$$

# Key Steps of Proof

- **Lemma (Energy Estimate)**

  *We have for* (0.25)

$$
\begin{aligned}
&\sup_{\frac{1}{2}\leq\bar{u}\leq\bar{U}}\left(\int_{\max(\bar{u},2-\bar{u})}^{+\infty}{\phi_u}^2 du\right)^{\frac{1}{2}} \\
&\lesssim\varepsilon\left(\|\psi_0\|^2_{H^1(\mathbb{R}^3)}+\|\psi_1\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}+\int_{\frac{1}{2}}^{\bar{U}}\left(\int_{\max(\bar{u},2-\bar{u})}^{+\infty}G^2 du\right)^{\frac{1}{2}}d\bar{u}.
\end{aligned}
\tag{0.26}
$$

# Key Steps of Proof

- **Lemma (Morawetz Type Estimate)**

  *We have for* (0.25)

$$\sup_{\frac{1}{2}\le\bar{u}\le\bar{U}}\left(\int_{\max(\bar{u},2-\bar{u})}^{+\infty}u^3(u-\bar{u}){\phi_u}^2du\right)^{\frac{1}{2}}$$

$$\lesssim\varepsilon\left(\|\psi_0\|^2_{H^1(\mathbb{R}^3)}+\|\psi_1\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}+\int_{\frac{1}{2}}^{\bar{U}}\left(\int_{\max(\bar{u},2-\bar{u})}^{+\infty}u^3(u-\bar{u})G^2du\right)^{\frac{1}{2}}d\bar{u}. \tag{0.27}$$

This can be proved by a multiplier $(u-\bar{u})\phi_u$ and establishing a Hardy type inequality

$$\int_{\max(\bar{u},2-\bar{u})}^{+\infty}\phi^2udu\lesssim\int_{\max(\bar{u},2-\bar{u})}^{+\infty}u^3{\phi_u}^2du.$$

# Key Steps of Proof

- **Lemma (Weighted $L^2 - L^2$ Estimate)**

  *We have for* (0.25)

$$\sup_{\frac{1}{2}\leq \bar{u}\leq \bar{U}} \left(\int_{\max(\bar{u},2-\bar{u})}^{+\infty} u^{3s}(u-\bar{u})^{s}{\phi_u}^{2}du\right)^{\frac{1}{2}}$$

$$\lesssim \varepsilon\left(\|\psi_0\|^2_{H^1(\mathbb{R}^3)}+\|\psi_1\|^2_{L^2(\mathbb{R}^3)}\right)^{\frac{1}{2}}+\int_{\frac{1}{2}}^{\bar{U}}\left(\int_{\max(\bar{u},2-\bar{u})}^{+\infty} u^{3s}(u-\bar{u})^{s}G^2du\right)^{\frac{1}{2}}d\bar{u}, \tag{0.28}$$

  *where* $0 \leq s \leq 1$.

This can be proved by interpolating between energy estimate and Morawetz type estimate.

# Key Steps of Proof

- Take $s = \frac{1}{4} + \frac{1}{2p}$ and denote

$$M(\phi)(\bar{u}) = \left( \frac{1}{2} \int_{\max(\bar{u}, 2-\bar{u})}^{+\infty} u^{3s} (u - \bar{u})^s {\phi_u}^2 du \right)^{\frac{1}{2}}.$$

We then show $M(\phi)(\bar{u})$ is bounded for $p > p_S(3 + \mu)$ and $3/2 \leq \mu < 2$.

Thank You for Your Attention!