On the effect of slowly decreasing initial data for nonlinear wave equations with damping and potential in the scaling critical regime

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Introduction

$$(1.1) \qquad (\partial_t^2 + D(r)\partial_t - \Delta + V(r))U = |U|^{p-1}U \quad \text{in } (0,T)\times\mathbb{R}^3,$$

$$(1.2) \qquad U(0,x) = \varepsilon f_0(x), \quad (\partial_t U)(0,x) = \varepsilon f_1(x) \quad \text{for } x \in \mathbb{R}^3,$$

$p > 1, \varepsilon > 0, r = |x|$

f_0, f_1 : *radially symmetric functions.*

Georgiev-K-Wakasa, JDE (2019)

$$V(r) = D(r)^2/4 - D'(r)/2 \quad \text{for } r > 0, \quad D(r) = 2/r \quad \text{for } r \geq 1.$$

Shift of critical exponent: $p_0(3) \Longrightarrow p_0(5)$

$$(\partial_t^2 - \Delta)U = |U|^{p-1}U \quad \text{in } (0,T) \times \mathbb{R}^n,$$
$$U(0,x) = \varepsilon f_0(x), \quad (\partial_t U)(0,x) = \varepsilon f_1(x) \quad \text{for } x \in \mathbb{R}^n,$$

$p_0(n)$ is the positive root of

$$\gamma(p,n) := 2 + (n+1)p - p^2 = 0, \ \ n \geq 2.$$

$p > p_0(n)$	Global existence for small initial data
$1 < p \leq p_0(n)$	Blow-up even for small initial data

Introduction

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$$(1.2) \qquad U(0,x) = \varepsilon f_0(x), \quad (\partial_t U)(0,x) = \varepsilon f_1(x) \quad \text{for } x \in \mathbb{R}^3,$$

$$p > 1, \varepsilon > 0, r = |x|$$

$$f_0, f_1 : \ \textit{radially symmetric functions.}$$

In this work

$$V(r) = D(r)^2/4 - D'(r)/2 \quad \text{for } r > 0, \quad D(r) = \mu/r \quad \text{for } r \geq 1.$$

$$\mu \geq 0$$

Introduction

$$(\partial_t^2 + D(r)\partial_t - \Delta + V(r))U = |U|^{p-1}U \quad \text{in } (0,T)\times\mathbb{R}^3, \tag{1.1}$$

$$U(0,x) = \varepsilon f_0(x), \quad (\partial_t U)(0,x) = \varepsilon f_1(x) \quad \text{for } x \in \mathbb{R}^3, \tag{1.2}$$

Dai-K-Sobajima, Nonlinear Anal. Real World Appl. (2021)

Lai-Liu-Tu-Wang, arXiv2021.10257 (2021)

Upper bound of the lifespan

Without the special relation between V(r) and D(r)

Initial data is compactly supported → Test function method

Introduction

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$$(1.2)\qquad U(0,x) = \varepsilon f_0(x), \quad (\partial_t U)(0,x) = \varepsilon f_1(x) \quad \text{for } x\in\mathbb{R}^3,$$

$$V(r) = D(r)^2/4 - D'(r)/2 \quad \text{for } r>0, \quad D(r) = \mu/r \quad \text{for } r \geq 1.$$

$$|f_0(r)| \leq (1+r)^{-\kappa}, \quad |f_0'(r)| + |f_1(r)| \leq (1+r)^{-\kappa-1}, \quad r>0$$

Critical exponent

$$p_c(\mu,\kappa) = \max\{p_0(3+\mu), 1+\frac{2}{\kappa}\}$$

Lifespan $T(\varepsilon)$

Maximal existence time of the corresponding integral equation

Results

Th. 1

Let $\kappa > \mu/2$. The initial data are radial and satisfy the decaying condition. If $p > p_0(3+\mu)$ and $p \geq 1 + \dfrac{2}{\kappa}$, then $T(\varepsilon) = \infty$ for sufficiently small ε.

Th. 2

Let $1 < p < p_0(3+\mu)$. If $f_0(r) \equiv 0,\ f_1(r) \geq 0\,(\not\equiv 0)$, then

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & (p = p_0(3+\mu)), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & (1 < p < p_0(3+\mu)). \end{cases}$$

Th. 3

Let $\kappa > 0$. Assume either $1 < p < 1 + \dfrac{2}{\kappa}$ or $p = 1 + \dfrac{2}{\kappa} = p_0(3+\mu)$. If $f_0(r) \equiv 0,\ f_1(r) \geq (1+r)^{-\kappa-1}$ then

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-(p-1)}) & (p = 1 + 2/\kappa = p_0(3+\mu)), \\ Cb(\varepsilon) & (1 < p < 1 + 2/\kappa \text{ and } \kappa = \mu/2 + 1 + 1/p), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \text{ and } \kappa \neq \mu/2 + 1 + 1/p). \end{cases}$$

Results

Th. 4

Under the same assumtion on Th.1 we have

$$T(\varepsilon) \geq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & (p = p_0(3+\mu) \ and \ \kappa > \mu/2 + 1 + 1/p), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & (1 < p < p_0(3+\mu) \ and \ \kappa > \mu/2 + 1 + 1/p), \\ \exp(C\varepsilon^{-(p-1)}) & (p = p_0(3+\mu) \ and \ \kappa = \mu/2 + 1 + 1/p), \\ Cb(\varepsilon) & (1 < p < p_0(3+\mu) \ and \ \kappa = \mu/2 + 1 + 1/p), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \ and \ \mu/2 < \kappa < \mu/2 + 1 + 1/p). \end{cases}$$

Here $b(\varepsilon)$ is the number defined by $\varepsilon^p b^{p(2-(p-1)\kappa)/(p-1)} \log(1+b) = 1$.

Results

Remark 1

$f_0(r) \equiv 0,\ f_1(r) \geq (1+r)^{-\kappa-1}$ → $f_0(r) \equiv 0,\ f_1(r) \geq 0 \,(\not\equiv 0)$

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-(p-1)}) & (p = 1 + 2/\kappa = p_0(3+\mu)), \\ Cb(\varepsilon) & (1 < p < 1 + 2/\kappa \ and \ \kappa = \mu/2 + 1 + 1/p), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \ and \ \kappa \neq \mu/2 + 1 + 1/p). \end{cases}$$

$$T(\varepsilon) \leq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & (p = p_0(3+\mu)), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & (1 < p < p_0(3+\mu)). \end{cases}$$

Remark 2

Under the same assumtion on Th.1 we have

$$T(\varepsilon) \geq \begin{cases} \exp(C\varepsilon^{-p(p-1)}) & (p = p_0(3+\mu) \ and \ \kappa > \mu/2 + 1 + 1/p), \\ C\varepsilon^{-2p(p-1)/\gamma(p,3+\mu)} & (1 < p < p_0(3+\mu) \ and \ \kappa > \mu/2 + 1 + 1/p), \\ \exp(C\varepsilon^{-(p-1)}) & (p = p_0(3+\mu) \ and \ \kappa = \mu/2 + 1 + 1/p), \\ Cb(\varepsilon) & (1 < p < p_0(3+\mu) \ and \ \kappa = \mu/2 + 1 + 1/p), \\ C\varepsilon^{-(p-1)/(2-(p-1)\kappa)} & (1 < p < 1 + 2/\kappa \ and \ \mu/2 < \kappa < \mu/2 + 1 + 1/p). \end{cases}$$

Key point

$$u(t,r) = rU(t, r\omega) \quad \text{with } r = |x|,\ \omega = x/|x|.$$

$$V(r) = D(r)^2/4 - D'(r)/2 \quad \text{for } r > 0, \quad D(r) = \mu/r \quad \text{for } r \geq 1.$$

$$(2.1) \qquad (\partial_t - \partial_r + w(r))(\partial_t + \partial_r + w(r))u = |u|^p/r^{p-1} \quad \text{in } (0,T) \times (0,\infty),$$

$$(2.2) \qquad u(0,r) = \varepsilon\varphi(r), \quad (\partial_t u)(0,r) = \varepsilon\psi(r) \quad \text{for } r > 0$$

$$w(r) = D(r)/2.$$

Key point

$$(2.4) \qquad E_-(t,r,y) = e^{-W(r)} e^{2W(2^{-1}(y-t+r))} e^{-W(y)} \quad \text{for } t, r \geq 0, \;\; y \geq t-r.$$

$$W(r) = \int_0^r w(\tau)d\tau \text{ for } r \geq 0,$$

$$(2.5) \qquad u(t,r) = \varepsilon u_0(t,r) + \frac{1}{2} \iint_{\Delta_-(t,r)} E_-(t-\sigma,r,y) \frac{|u(\sigma,y)|^p}{y^{p-1}} dy d\sigma$$

for $t > 0, \; r > 0,$ *where we have set*

$$(2.6) \qquad u_0(t,r) = \frac{1}{2} \int_{|t-r|}^{t+r} E_-(t,r,y) \left(\psi(y) + \varphi'(y) + w(y)\varphi(y) \right) dy + \chi(r-t) E_-(t,r,r-t) \varphi(r-t)$$

$$\Delta_-(t,r) = \{(\sigma,y) \in (0,\infty) \times (0,\infty); |t-r| < \sigma + y < t+r, \; \sigma - y < t-r\}$$

$$\chi(s) = 1 \text{ for } s \geq 0, \text{ and } \chi(s) = 0 \text{ for } s < 0.$$

Ingredients of the proof

Existence part: Light cone, Weighted L^∞ estimates

Blow-up part: Positivity, Lower bounds of solution

Future problems

Nonlinear scattering:

Remove the special relation:

$$V(r) = D(r)^2/4 - D'(r)/2 \quad \text{for } r > 0, \quad D(r) = \mu/r \quad \text{for } r \geq 1.$$

Thank you very much!!