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On the solvability of the synthesis problem for optimal control systems with distributed parameters

A. Kerimbekov

E. F. Abdyldaeva

A. A. Anarbekova

Kyrgyz-Russian Slavic University

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Annotation

The solvability of synthesis problem of external and boundary controls is investigated for optimization of oscillation process, described by partial differential equations with Fredholm integral operator. Functions of the external and boundary actions are nonlinearly with respect to control. An integro-differential equation is obtained in the specific type for Bellmann functional. An algorithm is developed for constructing solutions to synthesis problem of external and boundary controls.

Keywords: Generalized solution, Bellman functional, Frechet differential, integro-differential equation, Fredholm operator, optimal control synthesis.

Formulation of Synthesis Problem

Consider a controlled oscillation process described by following boundary value problem

$$v_{tt} - Av = \lambda \int_0^T K(t,\tau)v(\tau,x)d\tau + g(t,x)f[t,u(t)], \quad x \in Q, \quad 0 < t < T, \tag{1}$$

$$v(0,x) = \psi_1(x), \quad v_t(0,x) = \psi_2(x), \quad x \in Q, \tag{2}$$

$$\Gamma v(t,x) \equiv \sum_{i,k=1}^{n} a_{ik}(x)v_{x_k}(t,x)cos(\mu,x_i) + a(x)v(t,x) = \epsilon(t,x)p[t,\vartheta(t)], \tag{3}$$

$$x \in \gamma, 0 < t < T$$

Formulation of Synthesis Problem

where A is an elliptic operator defined by the formula

$$Av(t,x) = \sum_{i,k}^{n} (a_{ik} v_{x_k}(t,x))_{x_i} - c(x)v(t,x), \tag{4}$$

Q is domain of space R^n bounded by a piecewise smooth curve γ ;
$Q_T = Q \times [0, T)$; functions
$K(t,\tau) \in H(D), D = \{0 \leq t, \tau \leq T\}, \psi_1(x) \in H_1(Q), \psi_2(x) \in H(Q), a_{ik}(x),$
are known functions;μ is a normal vector, outgoing from the point $x \in \gamma$;
$f[t, u(t)] \in H(0, T)$, $\forall$ for external control $u(t) \in H(0, T)$,
$p[t, \vartheta(t)] \in H(0, T)$, $\forall$ boundary control $\vartheta(t) \in H(0, T), \gamma_T = \gamma \times (0, T)$;
$a(x) \geq 0$ and $c(x \geq 0)$ are measurable functions, $H(Y)$ is Hilbert space of square-summable functions defined on the set Y ; $H_1(Y)$ is Sobolev space of the first order; λ is a parameter; T is a fixed moment of time;

Functions of external and boundary influences are monotonic with regard to functional variable, i.e.

$$f_u[t, u(t)] \neq 0, \quad \forall t \in (0, T); \quad p_\vartheta[t, \vartheta(t)] \neq 0, \quad \forall\ t \in \gamma_T, \tag{5}$$

and $g(t, x) \in H(Q_T), \epsilon(t, x) \in H(\gamma_T)$ are given functions.
Note that, according to conditions (5) an one-to-one correspondence is established between the elements of the control space $\{[u(t), \vartheta(t)]\}$ and the space of controlled process states $\{v(t, x)\}$.
It is required to find such controls $u^0(t) \in H(0, T)$ and $\vartheta^0(t) \in H(0, T)$ that minimize the functional

$$J[u(t), \vartheta(t)] = \int_Q \{[v(T, x) - \xi_1(x)]^2 + [v_t(T, x) - \xi_2(x)]^2\} dx$$
$$+ \int_0^T \{\alpha M^2[t, u(t)] + \beta N^2[t, \vartheta(t)]\} dt,\ \ \alpha, \beta > 0, \tag{6}$$

$$\xi_1(x) \in H(Q), \xi_2(x) \in H(Q)$$

are given functions, defined on set of solutions of Boundary value problem (1)–(5).

The sought controls $u^0(t)$ and $\vartheta^0(t)$should be found as a function (functional) of the controlled process's state, i.e. as

$$u^0(t) = u[t, v(t, x), v_t(t, x)], \quad t \in (0, T),$$
$$\vartheta^0(t) = \vartheta[t, v(t, x), v_t(t, x)], \quad t \in (0, T).$$

Generalized solution of Boundary Value Problem

As it is known, in the study of applied control problems it is advisable to use the concept of a generalized solution of a boundary value problem.

Definition 2.1. A generalized solution of the boundary value problem (1)–(5) is a function $v(t,x) \in H(Q_T)$, that together with the generalized derivatives v_t and $v_{x_i}(t,x)$ satisfies the integral identity

$$\int_Q (v_t(t,x)\phi(t,x))_{t_1}^{t_2} dx = \int_{t_1}^{t_2} \{ \int_Q [v_t(t,x)\phi_t(t,x) - \sum_{i,k=1}^{n} a_{ik}(x) v_{x_k}(t,x)\phi_{x_i}(t,x)$$

$$-c(x)v(t,x)\phi(t,x) + (\lambda \int_0^T K(t,\tau)v(\tau,x)dx + g(t,x)f[t,u(t)])\phi(t,x)]dx$$

$$+ \int_\gamma (\epsilon(t,x)p[t,\vartheta(t)] - a(x)v(t,x))\phi(t,x)dx\} dt \quad (7)$$

for any t_1 and t_2 $(0 < t_1 \le t \le t_2 \le T)$ for any function $\phi(t,x) \in H_1(\overline{Q}_T)$,and satisfies the initial conditions in the weak sense, i.e. we have equalities

$$\lim_{t\to t_0} \int_Q [v(t,x) - \psi_1(x)]\phi_0(x)dx = 0, \quad \lim_{t\to t_0} \int_Q [v_t(t,x) - \psi_1(x)]\phi_1(x)dx = 0$$

for $\phi_0(x) \in H(Q)$ and $\phi_1(x) \in H(Q)$.
The solution of problem (1)–(5) we will seek in the form of

$$v(t,x) = \sum_{n=1}^{\infty} v_n(t)z_n(x), \quad v_n(t) = \int_Q v(t,x)z_n(x)dx, \qquad (8)$$

where $z_n(x)$ defines as a generalized eigenfunction of boundary value problem, for $n = 1, 2, 3, \ldots,$

$$B_n[\phi(t,x),z_j(x)] \equiv \int_Q \Big[\sum_{i,k=1}^{n} a_{ik}(x)\phi_{x_k}(t,x)z_{j_{xi}} + c(x)z_j(x)\phi(t,x)\Big]dx$$
$$+\int_\gamma a(x)z_j(x)\phi(t,x)dx = \lambda_j^2\int_Q \phi(t,x)z_j(x)dx,$$
$$\Gamma z_j(x) = 0, \quad x\in\gamma, \quad 0<t<T, \quad j=1,2,3,..., \qquad (9)$$

and they are form a complete orthonormal system in the Hilbert space $H(Q)$, and the corresponding eigenvalues λ_j satisfy the conditions
$\lambda_j \le \lambda_{j+1}, j = 1,2,3,..., \lim \lambda_j = \infty$ at $j\to\infty$.
Using Liouville's method,we find the Fourier coefficients $v_n(t)$ for each fixed $n = 1,2,3,...,$ as a solution of a linear Fredholm integral equation of the second kind of the following form

$$v_n(t) = \lambda \int_0^T K_n(t,s) v_n(s) ds + a_n(t), \tag{10}$$

where

$$K_n(t,s) = \frac{1}{\lambda_n} \int_0^t sin\lambda_n(t-\tau) K(\tau,s) d\tau, \quad K_n(0,s) = 0, \quad n = 1,2,3,...,$$

$$a_n(t) = \psi_{1n} cos\lambda_n t + \frac{1}{\lambda_n} \psi_{2n} sin\lambda_n t$$

$$+\frac{1}{\lambda_n} \int_0^t sin\lambda_n(t-\tau)(g_n(\tau) f[\tau, u(\tau)] + \epsilon_n(\tau) p[\tau, \vartheta(\tau)]) d\tau, \tag{11}$$

$$g_n[\tau] = \int_Q g[\tau, x] z_n(x) dx, \quad \epsilon_n[\tau] = \int_\gamma \epsilon[\tau, x] z_n(x) dx. \tag{12}$$

The solution to integral equation (10) is found by formula

$$v_n(t) = \lambda \int_0^T R_n(t, s, \lambda) a_n(s) ds + a_n(t), \tag{13}$$

where $R_n(t, s, \lambda)$ is the resolvent of kernel $K_n(t, s)$. The resolvent is a continuous function for the values of the parameter λ satisfying the following estimates for any $n = 1, 2, 3, \ldots,$

$$|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}}, \tag{14}$$

where

$$K_0 = \int_0^T \int_0^T K^2(t,\tau)d\tau dt,$$

$$\int_0^T R_n(t,s,\lambda)ds \leq \frac{K_0 T}{(\lambda_n - |\lambda| T\sqrt{K_0})^2}. \tag{15}$$

Thus, the solution to the boundary value problem (1)–(5)

$$v(t,x) = \sum_{n=1}^{\infty} \left[\lambda \int_0^T R_n(t,s,\lambda)a_n(s)ds + a_n(t)\right] z_n(x) \tag{16}$$

differentiating with respect to t, we obtain the

$$v_t(t,x) = \sum_{n=1}^{\infty} \left[\lambda \int_0^T R'_{nt}(t,s,\lambda) a_n(s) ds + a'_n(t) \right] z_n(x).$$

Taking into account (11)–(15) and inequality

$$\int_0^T R_{nt}^2(t,s,\lambda) ds \leq \frac{TK_0 \lambda_n^2}{(\lambda_n - |\lambda| T\sqrt{K_0})^2}. \qquad (17)$$

Based on these calculations, one can prove $v(t,x), v_t(t,x) \in H(Q_T)$

On Solvability of the Synthesis Problem

For functional (6), Bellman functional is defined in the form

$$S[t,\omega(t,x)] = \min_{u\in U,\vartheta\in V}\Big\{\int_t^T \{\alpha M^2[\tau,u(\tau)]$$

$$+\beta N^2[\tau,\vartheta(\tau)]\}d\tau + \int_Q \parallel \omega(T,x)-\xi(x)\parallel^2 dx\Big\}, \qquad (18)$$

where $\omega(t,x)=\{v(t,x),v_t(t,x)\}$ is a vector function of states; $\xi(x)=\{\xi_1(x),\xi_2(x)\}$ is a vector function of the desired state of the controlled process at the moment of time T; $\parallel . \parallel$ is a norm of vector; U is a set of allowed values of control $u(t), t\in(0,T)$;V is a set of allowed values of control $\vartheta(t), t\in(0,T)$. According to the Bellman-Egorov scheme, assuming that $S[t,\omega(t,x)]$ as differentiable function with respect to t and as a Frechet differentiable functional can be rewritten as

$$-\frac{\partial S[t,\omega(t,x)]}{\partial t}\Delta t = \min_{u\in U,\vartheta\in V}\left\{\int_t^{t+\Delta t}\Big(\alpha M^2[\tau,u(\tau)] + \beta N^2[\tau,\vartheta(\tau)]\Big)d\tau + ds[t,\omega(t,x);\Delta\omega(t,x)] + o(\Delta t) + \delta[t,\omega(t,x);\Delta\omega(t,x)]\right\}, \qquad (19)$$

where $\Delta\omega(t,x) = \omega[t+\Delta t,x] - \omega[t,x]$, $ds[t,\omega(t,x);\Delta\omega(t,x)]$ is a Frechet differential, $\delta[t,\omega(t,x);\Delta\omega(t,x)]$ sre infinitesimal values with respect to Δt. As Frechet differential is linear functional with respect to $\Delta\omega(t,x) \in H^2(Q_T) = H(Q_T)\times H(Q_T)$, $\forall(t,x)\in Q_T$, the following equality is hold

$$ds[t,\omega(t,x);\Delta\omega(t,x)] = \int_Q m^*(t,x)\Delta\omega(t,x)dx \equiv \int_Q \Big\{ m_1(t,x)\Delta v(t,x) + m_2(t,x)\Delta v_t(t,x) \Big\} dx, \quad (20)$$

where * is a transpose symbol; Vector -function $m(t,x) = \{m_1(t,x), m_2(t,x)\}$ is a gradient of functional $S[t,\omega(t,x)]$ and belongs to space $H^2(Q_T)$ in almost all $(t,x) \in Q_T$. Note, that $m(t,x)$ is defined depending on the functional $S[t,\omega(t,x)]$, i.e.

$$m(t,x) = m(t,x,S[t,\omega(t,x)]). \quad (21)$$

the following identity is hold

$$\int_Q m^*(t,x)\Delta\omega(t,x)dx = \int_Q (m_2(t,x)\Delta v_t(t,x))_t^{t+\Delta t}dx$$

$$+\int_Q m_1(t,x)\Delta v(t,x)dx - \int_Q \Delta m_2(t,x)v_t(t+\Delta t,x)dx. \quad (22)$$

taking (20)–(22) into account equality (19) can be rewritten into equality in form of

$$-\frac{\partial S[t,\omega(t,x)]}{\partial t}\Delta t = \min_{u\in U,\vartheta\in V}\Big\{\int_t^{t+\Delta t}\Big[\alpha M^2[\tau,u(\tau)]$$

$$+\beta N^2[\tau,\vartheta(\tau)]\Big]d\tau + \int_Q (m_2(\tau,x)v_t(\tau,x))_t^{t+\Delta t}dx + \int_Q [m_1(t,x)\Delta v(t,x)$$

$$-\Delta m_2(t,x)v_t(t+\Delta t,x)]dx + o(\Delta t) + \delta[t,\omega(t,x);\Delta\omega(t,x)]\Big\}. \quad (23)$$

Let $m_2(t,x) \in H_1(Q_T)$. Then in the integral identity (7) assuming that $\phi(t,x) \equiv m_2(t,x)$ and $t_1 = t$, $t_2 = t + \Delta t$ we have

$$\int_Q (m_2(\tau,x)v_t(\tau,x))_t^{t+\Delta t} dt \equiv \int_t^{t+\Delta t} \Big\{ \int_Q \Big[m_{2t}(\tau,x)v_t(\tau,x)$$

$$- \sum_{i,k=1}^{n} a_{ik}(x) v_{x_k}(\tau,x) m_{2x_i}(\tau,x) - c(x)v(\tau,x)m_2(\tau,x)$$

$$+ \left(\lambda \int_0^T K(\tau,\sigma)v(\sigma,x)d\sigma + g(\tau,x)f[\tau,u(\tau)] \right) m_2(\tau,x) \Big] dx$$

$$+ \int_\gamma \Big(\epsilon(\tau,x)p[\tau,\vartheta(\tau)] - a(x)v(\tau,x) \Big) m_2(\tau,x)dx \Big\} d\tau.$$

Taking this identity into account, we rewrite the equality (23) in the form

$$
\begin{aligned}
-\frac{\partial S[t,\omega(t,x)]}{\partial t} = \min_{u\in U,\vartheta\in V}\Bigg\{\frac{1}{\Delta t}\int_t^{t+\Delta t}\Big[\alpha M^2[\tau,u(\tau)]+\beta N^2[\tau,\vartheta(\tau)]\Big]d\tau \\
+\frac{1}{\Delta t}\int_t^{t+\Delta t}\Bigg(\int_Q\Big[m_{2t}(\tau,x)v_t(\tau,x)-\sum_{i,k=1}^{n}a_{ik}(x)v_{x_k}(\tau,x)m_{2x_i}(\tau,x) \\
-c(x)v(\tau,x)m_2(\tau,x)+\Big(\lambda\int_0^T K(\tau,\sigma)v(\sigma,x)d\sigma+g(\tau,x)f[\tau,u(\tau)]\Big)m_2(\tau,x)\Big]dx \\
+\int_\gamma\Big(\epsilon(\tau,x)p[\tau,\vartheta(\tau)]-a(x)v(\tau,x)\Big)m_2(\tau,x)dx\Bigg)d\tau \\
+\int_Q\Big[m_1(t,x)\frac{\Delta v(t,x)}{\Delta t}-\frac{\Delta m_2(t,x)}{\Delta t}v_t(t+\Delta t,x)\Big]dx+ \\
\frac{o(\Delta t)}{\Delta t}+\frac{\delta[t,\omega(t,x);\Delta\omega(t,x)]}{\Delta t}\Bigg\}.
\end{aligned}
$$

According the following relation

$$\lim_{t\to+0}\frac{o(\Delta t)}{\Delta t}=0,\quad \lim_{t\to+0}\frac{\delta[t,\omega;\Delta\omega]}{\Delta t}=0,$$

we obtain nonlinear integro-differential equation of Bellman-type

$$-\frac{\partial S[t,\omega(t,x)]}{\partial t}=\min_{u\in U,\vartheta\in Y}\Big\{\alpha M^2[t,u(t)]+\int_Q m_2(t,x)g(t,x)f[t,u(t)]dx$$

$$+\beta N^2[t,\vartheta(t)]+\int_\gamma m_2(t,x)\epsilon(t,x)p[t,\vartheta(t)]dx+\int_Q\Big(\lambda\int_0^T K(t,\tau)v(\tau,x)d\tau\Big)m_2(t,x)dx$$

$$+\int_Q m_1(t,x)v_t(t,x)dx-\int_Q\Big[\sum_{i,k=1}^n a_{ik}(x)v_{x_k}(t,x)m_{2_{x_i}}(t,x)$$

$$+c(x)v(t,x)m_2(t,x)\Big]dx-\int_\gamma a(x)v(t,x)m_2(t,x)dx\Big\}. \qquad (24)$$

According to (18) this equation should be considered with the condition

$$S[T,\omega(T,x)] = \int_Q \| \omega(T,x) - \xi(x) \|^2 \, dx \qquad (25)$$

Thus, function $S[t,\omega(t,x)]$ should be found as a solution to problem (24)—(25), which is called the Cauchy-Bellman problem. To solve this problem, we first solve the minimization problem in the right-hand side of the equation (24). In this case, we have the following cases:

1. U and V —are open sets;
2. U — is an open set ,V — is a closed set;
3. U — is a closed set, V — is an open set;
4. U and V — are closed sets .

Consider the first case:
Using the classical method for solving the extremum problem, we find that the "controls suspicious for optimality" , control $u^0(t)$ is defined as follows:
The desired control $u^0(t)$ is determined according to the optimality conditions in the form of the equality

$$2\alpha M[t,u(t)]M_u[t,u(t)] + \int_Q m_2(t,x)g(t,x)dxf_u[t,u(t)] = 0 \qquad (26)$$

and differential inequality

$$2\alpha\Big(M[t,u(t)]M_u[t,u(t)]\Big)_u + \int_Q m_2(t,x)g(t,x)dxf_{uu}[t,u(t)] > 0$$

Which are fulfilled simultaneously for almost all $(t,x) \in Q_T$. Differential inequality is a difficult condition to verify. However, it can be transformed to the form of

$$f_u[t,u(t)]\Big(\frac{M[t,u(t)]M_u[t,u(t)]}{f_u[t,u(t)]}\Big)_u > 0. \qquad (27)$$

Let the optimality conditions (26) and (27) be satisfied. Then, according to the implicit function theorem from equality (26), the control $u(t)$ is uniquely determined, that is, there exists a function φ such that

$$u^0(t) = \varphi_1[t, \int_Q m_2(t,x)g(t,x)dx, \alpha] \quad t \in (0,T). \qquad (28)$$

Similarly, the "controls suspicious for optimality" , boundary control $\vartheta^0(t)$ is defined as follows: According to the optimality conditions in the form of the equality

$$2\beta N[t,\vartheta(t)]N_\vartheta[t,\vartheta(t)] + \int_\gamma m_2(t,x)\epsilon(t,x)p[t,\vartheta(t)]dx = 0 \qquad (29)$$

and differential inequality

$$p_{\vartheta}[t,\vartheta(t)]\Big(\frac{N[t,\vartheta(t)]N_{\vartheta}[t,\vartheta(t)]}{p_{\vartheta}[t,\vartheta(t)]}\Big)_{\vartheta}>0. \tag{30}$$

the desired control $\vartheta^0(t)$ is determined by following formula

$$\vartheta^0(t)=h_1[t,\int_{\gamma}m_2(t,x)\epsilon(t,x)dx,\beta].\quad t\in(0,T).$$

Conclusion

This article shows some features of the considered synthesis problem, in particular, the presence of the Fredholm integral operator in the boundary value problem significantly affects the solvability of the Cauchy-Bellman problem, and as well as on the construction of an algorithm for synthesizing controls depending on the state of the controlled process. The results obtained can be used in the development of new research methods and methods for solving nonlinear synthesis problems.

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Thank You For Attention!