On pointwise convergence of Jacobi-Dunkl series

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Jacobi-Dunkl polynomials

Notation

Let $a \in \mathbb{C}$. We denote by

$$(a)_n := \begin{cases} a(a+1)...(a+n-1) & \text{if} \quad n \in \mathbb{N} \setminus \{0\}, \\ 1 & \text{if} \quad n = 0. \end{cases}$$

$(a)_n$ is called Pochhammer symbol.

Definition

Let $a, b \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \mathbb{Z}_-$.
The Gauss hypergeometric function ${}_2F_1(a, b; c; .)$ is defined by

$$\forall\, z \in \mathbb{C};\ |z| < 1,\ {}_2F_1(a, b; c; z) := \sum_{n=0}^{+\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n.$$

In the sequel of this talk, we consider $\alpha \geq \beta \geq -\frac{1}{2};\ \alpha \neq -\frac{1}{2}$
and $\rho := \alpha + \beta + 1$.

Definition [1]

The normalized Jacobi polynomials $\varphi_m^{(\alpha,\beta)}(\theta)$ are defined by

$$\varphi_m^{(\alpha,\beta)}(\theta) := R_m^{(\alpha,\beta)}(\cos(2\theta)) = {}_2F_1\left(-m, m+\rho; \alpha+1; (\sin\theta)^2\right),$$

$$m \in \mathbb{N},\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Definition [1]

The Jacobi operator $\Delta_{\alpha,\beta}$ defined on $C^2\left(\left(0, \frac{\pi}{2}\right)\right)$ is given by

$$\Delta_{\alpha,\beta} := \frac{d^2}{d\theta^2} + \frac{A'_{\alpha,\beta}}{A_{\alpha,\beta}}\frac{d}{d\theta},$$

where

$$A_{\alpha,\beta}(\theta) := \begin{cases} 2^{2\rho}(\sin|\theta|)^{2\alpha+1}(\cos\theta)^{2\beta+1} & \text{if} \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}, \\ 0 & \text{if} \quad \theta = 0. \end{cases}$$

Proposition [1]

For all $m \in \mathbb{N}$, $\varphi_m^{(\alpha,\beta)}$ is the unique even $\mathcal{C}^\infty$-solution on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ of the differential equation

$$\begin{cases} \Delta_{\alpha,\beta} u &= -\left(\lambda_m^{\alpha,\beta}\right)^2 u, \\ u(0) &= 1, \\ u'(0) &= 0, \end{cases}$$

with $\lambda_n^{(\alpha,\beta)} := 2\mathrm{sgn}(n)\sqrt{|n|(|n|+\rho)}, \quad n \in \mathbb{Z}$.

Definition [1]

The Jacobi-Dunkl operator is the differential-difference operator $\wedge_{\alpha,\beta}$ acting by

$$\wedge_{\alpha,\beta} f(\theta) := \frac{d}{d\theta} f(\theta) + \frac{A'_{\alpha,\beta}(\theta)}{A_{\alpha,\beta}(\theta)} \frac{f(\theta) - f(-\theta)}{2}, \quad f \in C^1\left(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right),$$

where

$$\frac{A'_{\alpha,\beta}(\theta)}{A_{\alpha,\beta}(\theta)} = (2\alpha+1)\cot\theta - (2\beta+1)\tan\theta\,, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}.$$

Theorem [1]

For each $n \in \mathbb{Z}$, the problem

$$\begin{cases} \wedge_{\alpha,\beta} u(\theta) &= i\lambda_n^{(\alpha,\beta)} u(\theta), \\ u(0) &= 1, \end{cases}$$

admits a unique $\mathcal{C}^\infty$-solution on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, denoted by $\psi_n^{(\alpha,\beta)}(\theta)$ called Jacobi-Dunkl polynomial and it is given by

$$\psi_n^{(\alpha,\beta)}(\theta) := \begin{cases} \varphi_{|n|}^{(\alpha,\beta)}(\theta) + \dfrac{i\lambda_n^{(\alpha,\beta)}}{4(\alpha+1)} \sin(2\theta) \varphi_{|n|-1}^{(\alpha+1,\beta+1)}(\theta) & \text{if} \quad n \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if} \quad n = 0, \end{cases}$$

which can be also expressed as follows :

$$\psi_n^{(\alpha,\beta)}(\theta) = \begin{cases} \varphi_{|n|}^{(\alpha,\beta)}(\theta) - \dfrac{i}{\lambda_n^{(\alpha,\beta)}} \dfrac{d}{d\theta} \varphi_{|n|}^{(\alpha,\beta)}(\theta) & \text{if} \quad n \in \mathbb{Z} \setminus \{0\}, \\ 1 & \text{if} \quad n = 0. \end{cases}$$

Proposition [1]

1. $\psi_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(\theta) = e^{2in\theta}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$
2. For all $n \in \mathbb{Z}, \ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have
$$\psi_{-n}^{(\alpha,\beta)}(\theta) = \psi_n^{(\alpha,\beta)}(-\theta) = \overline{\psi_n^{(\alpha,\beta)}(\theta)}.$$
3. Let $n \in \mathbb{Z}$, $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. We have
$$\psi_{2n}^{(\alpha,\alpha)}(\theta) = \psi_n^{(\alpha,-\frac{1}{2})}(2\theta).$$
4. For all $n \in \mathbb{Z}$, $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we have
$$\left|\psi_n^{(\alpha,\beta)}(\theta)\right| \leq 1.$$

Proposition [1]

For all n, $p \in \mathbb{Z}$, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_n^{(\alpha,\beta)}(\theta)\overline{\psi_p^{(\alpha,\beta)}(\theta)}A_{\alpha,\beta}(\theta)\,d\theta = \left(h_n^{(\alpha,\beta)}\right)^{-1}\delta_{np},$$

with

$$h_n^{(\alpha,\beta)} := \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\psi_n^{(\alpha,\beta)}(\theta)\right|^2 A_{\alpha,\beta}(\theta)\,d\theta\right)^{-1},$$

or even

$$h_n^{(\alpha,\beta)} = \begin{cases} \dfrac{(2|n|+\rho)\Gamma(|n|+\alpha+1)\Gamma(|n|+\rho)}{2^{2\rho+1}(\Gamma(\alpha+1))^2\Gamma(|n|+1)\Gamma(|n|+\beta+1)} & \text{if} \quad n \in \mathbb{Z}\setminus\{0\}, \\[2ex] \dfrac{\Gamma(\rho+1)}{2^{2\rho}\Gamma(\alpha+1)\Gamma(\beta+1)} & \text{if} \quad n = 0. \end{cases}$$

Theorem (Christoffel-Darboux Formula) [3]

$\forall\, n \in \mathbb{Z} \setminus \{0\},\ \forall\, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ \forall\, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];\ \theta \neq \pm\phi,$

$$\sum_{p=-n}^{n} \psi_p^{(\alpha,\beta)}(\theta)\overline{\psi_p^{(\alpha,\beta)}}(\phi) h_p^{(\alpha,\beta)} = \frac{\Gamma(\alpha+n+2)\Gamma(\rho+n+1)}{2^{2\rho-1}\left(\Gamma(\alpha+1)\right)^2 n!\Gamma(\beta+n+1)}$$

$$\times \frac{1}{\cos(2\theta)-\cos(2\phi)} \Big[\varphi_{n+1}^{(\alpha,\beta)}(\theta)\varphi_n^{(\alpha,\beta)}(\phi) - \varphi_n^{(\alpha,\beta)}(\theta)\varphi_{n+1}^{(\alpha,\beta)}(\phi) + \frac{\lambda_n\lambda_{n+1}}{4(n+1)(n+\rho)}$$

$$\Im\psi_{n+1}(\theta)\Im\psi_n^{(\alpha,\beta)}(\phi) - \Im\psi_n^{(\alpha,\beta)}(\theta)\Im\psi_{n+1}(\phi)\Big],$$

with

$$\Im\psi_n^{(\alpha,\beta)}(\theta) := \frac{\psi_n(\theta)-\psi_n(-\theta)}{2i}.$$

Proposition [2]

For all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have

$$\varphi_n^{(\alpha,\beta)}(\theta) \underset{+\infty}{\sim} \frac{2^{\rho}\Gamma(\alpha+1)n^{-\left(\alpha+\frac{1}{2}\right)}}{\sqrt{\pi}A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)} \cos\left[(2n+\rho)\theta-(2\alpha+1)\frac{\pi}{4}\right]. \quad (1)$$

$$\Im\psi_n^{(\alpha,\beta)}(|\theta|) \underset{+\infty}{\sim} \frac{2^{2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}} \frac{|n|^{-\left(\alpha+\frac{1}{2}\right)}}{A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)} \sin\left[(2|n|+\rho)|\theta|-(2\alpha+1)\frac{\pi}{4}\right]. \quad (2)$$

Continuous Jacobi-Dunkl convolution

Notation [6]

$$G_{\alpha,\beta} := \begin{cases} \mathbb{R}\backslash\{n\pi\}_{n\in\mathbb{Z}} & \text{if} \quad \alpha > \beta \geq -\frac{1}{2}, \\ \mathbb{R}\backslash\left\{\frac{n\pi}{2}\right\}_{n\in\mathbb{Z}} & \text{if} \quad \alpha = \beta \geq -\frac{1}{2}, \\ \emptyset & \text{if} \quad \alpha = \beta = -\frac{1}{2}. \end{cases}$$

Theorem [6]

Let $\theta, \phi \in G_{\alpha,\beta}$ and $k \in \mathbb{Z}$. We have

$$\psi_k^{(\alpha,\beta)}(\theta)\psi_k^{(\alpha,\beta)}(\phi) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi_k^{(\alpha,\beta)}(\varphi) W(\theta,\phi,\varphi) A_{\alpha,\beta}(\varphi)\, d\varphi,$$

where the explicit expression of the function W is given in [6].

Notation [4, 5]

Let $p \in [1, +\infty]$. We denote by

$L^p_{\alpha,\beta} := L^p\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right], A_{\alpha,\beta}(\theta)\, d\theta\right)$: the space of measurable functions f on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that

$$\begin{cases} \|f\|_{p,\alpha,\beta} := \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |f(\theta)|^p A_{\alpha,\beta}(\theta)\, d\theta\right)^{\frac{1}{p}} < +\infty & \text{if} \quad p \in [1, +\infty), \\ \|f\|_{\infty,\alpha,\beta} := \operatorname*{ess\,sup}_{\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} |f(\theta)| < +\infty & \text{if} \quad p = +\infty. \end{cases}$$

Definition [4, 6]

Let $\theta, \phi \in \mathbb{R}$ and $f \in L^p_{\alpha,\beta}, \quad p \in [1, +\infty]$.
The Jacobi-Dunkl translation operator $\tau^\phi f$ is defined by

$$\tau^\phi f(\theta) := \begin{cases} \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\varphi) W(\theta, \phi, \varphi) A_{\alpha,\beta}(\varphi)\, d\varphi & \text{if} \quad \theta, \phi \in G_{\alpha,\beta}, \\ f(\theta + \phi) & \text{if} \quad \theta \notin G_{\alpha,\beta} \quad \text{or} \quad \phi \notin G_{\alpha,\beta}. \end{cases}$$

Proposition [4, 6]

For all $p \in [1, +\infty]$, we have
If $f \in L^p_{\alpha,\beta}$, then $\tau^\phi f \in L^p_{\alpha,\beta}$.
And,

$$\|\tau^\phi f\|_{p,\alpha,\beta} \leq 2^{\left|1 - \frac{2}{p}\right|} \|f\|_{p,\alpha,\beta}.$$

Definition [4]

Let $f, g \in L^1_{\alpha,\beta}$. The generalized convolution product of f and g is defined by

$$f\#g(\theta) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tau^{-\phi} f(\theta) g(\phi) A_{\alpha,\beta}(\phi)\, d\phi, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Proposition [4]

Let $f, g, h \in L^1_{\alpha,\beta}$, we have

1. $f\#g = g\#f$.
2. $(f\#g)\#h = f\#(g\#h)$.

Theorem [4]

Let $f \in L^1_{\alpha,\beta}$ and $g \in L^p_{\alpha,\beta}$, $\quad p \in [1, +\infty]$. We have

$$\|f\#g\|_{p,\alpha,\beta} \leq 2\|f\|_{1,\alpha,\beta}\|g\|_{p,\alpha,\beta}.$$

Jacobi coefficients

Notation [1]

We denote by
$\widetilde{L}^p_{\alpha,\beta} := L^p\left(\left[0, \frac{\pi}{2}\right], A_{\alpha,\beta}(\theta)\, d\theta\right)$ the space of measurable functions g on $\left[0, \frac{\pi}{2}\right]$ such that

$$\begin{cases} \left(\int_0^{\frac{\pi}{2}} |g(\theta)|^p A_{\alpha,\beta}(\theta)\, d\theta\right)^{\frac{1}{p}} < +\infty & \text{if} \quad p \in [1, +\infty), \\ \underset{\theta \in \left[0, \frac{\pi}{2}\right]}{\operatorname{ess\,sup}} |g(\theta)| < +\infty & \text{if} \quad p = +\infty. \end{cases}$$

Definition [1]

The Jacobi coefficients of a function $g \in \widetilde{L}^1_{\alpha,\beta}$ are defined by

$$\forall\, m \in \mathbb{N}, \quad \mathcal{F}_{\alpha,\beta}(g)(m) = \int_0^{\frac{\pi}{2}} g(\theta)\varphi_m^{(\alpha,\beta)}(\theta) A_{\alpha,\beta}(\theta)\, d\theta.$$

Jacobi-Dunkl coefficients

Definition [1, 4, 5]

The Jacobi-Dunkl coefficients of a function $f \in L^1_{\alpha,\beta}$ are defined by

$$\mathcal{F}f(n) := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\overline{\psi_n^{(\alpha,\beta)}(\theta)}A_{\alpha,\beta}(\theta)\,d\theta, \quad n \in \mathbb{Z}.$$

Proposition [4]

Let $f \in L^1_{\alpha,\beta}$ and $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We have

$$\forall\, k \in \mathbb{Z}, \quad \mathcal{F}\left(\tau^{\theta} f\right)(k) = \psi_k^{(\alpha,\beta)}(\theta)\mathcal{F}f(k).$$

Proposition [4]

Let $f, g \in L^1_{\alpha,\beta}$. We have

$$\mathcal{F}(f\#g)(k) = \mathcal{F}f(k)\mathcal{F}(g)(k), \quad k \in \mathbb{Z}.$$

Notation [1, 4]

We denote by

$$l_h^1 := \left\{ a := (a_n)_{n\in\mathbb{Z}} : \mathbb{Z} \longrightarrow \mathbb{C} \ ; \ \|a_n\|_1 := \sum_{n=-\infty}^{+\infty} |a_n| h_n^{(\alpha,\beta)} < +\infty \right\}.$$

Theorem [1]

Let $f \in L^1_{\alpha,\beta}$ such that $\mathcal{F}f \in l_h^1$. Then,

$$f(\theta) = \sum_{n=-\infty}^{+\infty} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)h_n^{(\alpha,\beta)}, \ a.e \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Proposition [1]

For all $f \in L^2_{\alpha,\beta}$ (resp. $\in \widetilde{L}^2_{\alpha,\beta}$), we have

$$\mathcal{F}f(n) = o\left(|n|^{-\left(\alpha+\frac{1}{2}\right)}\right), \ n \to \infty \ \left(\text{resp.}\, \mathcal{F}_{\alpha,\beta}(f)(m) = o\left(m^{-\left(\alpha+\frac{1}{2}\right)}\right), \ m \to \infty\right).$$

Trigonometric Jacobi-Dunkl coefficients

Notations [5]

Let $f \in L^1_{\alpha,\beta}$. For all $k \in \mathbb{N}$, we put

$$a_k(f) := \mathcal{F}f(k) + \mathcal{F}f(-k),$$

and

$$b_k(f) := \begin{cases} -\dfrac{i}{\lambda_k^{(\alpha,\beta)}}\left[\mathcal{F}f(k) - \mathcal{F}f(-k)\right] & \text{if} \quad k \in \mathbb{N} \setminus \{0\}, \\ 0 & \text{if} \quad k = 0. \end{cases}$$

Remark [5]

For all $k \in \mathbb{N}$, we have these relations :

1. $\mathcal{F}f(k) = \dfrac{a_k(f) + i\lambda_k^{(\alpha,\beta)} b_k(f)}{2}$.
2. $\mathcal{F}f(-k) = \dfrac{a_k(f) - i\lambda_k^{(\alpha,\beta)} b_k(f)}{2}$.

Proposition [5]

For all $k \in \mathbb{N}$, we have the following integral representations :

1. $a_k(f) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\varphi_k^{(\alpha,\beta)}(\theta)A_{\alpha,\beta}(\theta)\, d\theta.$
2. $b_k(f) = \dfrac{2}{\left(\lambda_k^{(\alpha,\beta)}\right)^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\theta)\dfrac{d}{d\theta}\varphi_k^{(\alpha,\beta)}(\theta)A_{\alpha,\beta}(\theta)\, d\theta, \quad k \neq 0.$

Remarks [5]

Let $k \in \mathbb{N}$.

1. If the function f is even, then

$$b_k(f) = 0 \quad \text{and} \quad a_k(f) = 4\int_0^{\frac{\pi}{2}} f(\theta)\varphi_k^{(\alpha,\beta)}(\theta)A_{\alpha,\beta}(\theta)\, d\theta.$$

2. If the function f is odd, then

$$a_k(f) = 0, \quad b_k(f) = \frac{4}{\left(\lambda_k^{(\alpha,\beta)}\right)^2}\int_0^{\frac{\pi}{2}} f(\theta)\frac{d}{d\theta}\varphi_k^{(\alpha,\beta)}(\theta)A_{\alpha,\beta}(\theta)\, d\theta, \quad k \neq 0.$$

Proposition [5]

Let f be in $L^1_{\alpha,\beta}$, a real-valued function. For all $k \in \mathbb{N}$, we have the following properties :

1. $\mathcal{F}f(-k) = \overline{\mathcal{F}f(k)}$.
2. $a_k(f) = 2\Re\left(\mathcal{F}f(k)\right) \in \mathbb{R}$.
3. $b_k(f) = \dfrac{2}{\lambda_k^{(\alpha,\beta)}}\Im\left(\mathcal{F}f(k)\right) \in \mathbb{R}, \quad k \neq 0$.

Pointwise convergence of Jacobi-Dunkl series : Dirichlet theorem

Definition [1, 5]

We define the Jacobi-Dunkl series as

$$\sum_{n=-\infty}^{+\infty} \mathcal{F}f(n)\psi_n^{(\alpha,\beta)}(\theta)h_n^{(\alpha,\beta)}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

And, for $m \in \mathbb{N}$,

$$S_m^f(\theta) := \sum_{k=-m}^{m} \mathcal{F}f(k)\psi_k^{(\alpha,\beta)}(\theta)h_k^{(\alpha,\beta)}, \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

denotes its m^{th} partial sum.

Notation [5]

For all $n \in \mathbb{N}$, $\theta, \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We denote by

$$D_n^{(\alpha,\beta)}(\theta,\phi) := \sum_{k=-n}^{n} \psi_k^{(\alpha,\beta)}(\theta)\overline{\psi_k^{(\alpha,\beta)}(\phi)}h_k^{(\alpha,\beta)}.$$

$D_n^{(\alpha,\beta)}(\theta,\phi)$ is the analog of the Dirichlet kernel associated with the Fourier series.

Proposition [5]

Let $f \in L^1_{\alpha,\beta}$, $n \in \mathbb{N}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We have

$$S_n^f(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\phi) D_n^{(\alpha,\beta)}(\theta,\phi) A_{\alpha,\beta}(\phi)\, d\phi.$$

Theorem [5]

Let f be a piecewise continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$ such that

i) $f(-\theta) = f(\theta)$,

ii) f is differentiable on θ and $-\theta$.

Then, we have

$$\lim_{n \to +\infty} S_n^f(\theta) = f(\theta).$$

Proof : [5]

Let $n \in \mathbb{N}$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$. We can write

$$f(\theta) - S_n^f(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [f(\theta) - f(\phi)] D_n^{(\alpha,\beta)}(\theta,\phi) A_{\alpha,\beta}(\phi)\, d\phi.$$

From the Christoffel-Darboux formula, we have for all $\theta \neq \pm\phi$

$$\begin{aligned} f(\theta) - S_n^f(\theta) &= I_n^{(\alpha,\beta)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\theta) - f(\phi)}{\cos(2\theta) - \cos(2\phi)} \\ &\times \left[\varphi_{n+1}^{(\alpha,\beta)}(\theta)\varphi_n^{(\alpha,\beta)}(\phi) - \varphi_n^{(\alpha,\beta)}(\theta)\varphi_{n+1}^{(\alpha,\beta)}(\phi) \frac{\lambda_n^{(\alpha,\beta)}\lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \right. \\ &\times \left. \left(\Im\psi_{n+1}^{(\alpha,\beta)}(\theta)\Im\psi_n^{(\alpha,\beta)}(\phi) - \Im\psi_n^{(\alpha,\beta)}(\theta)\Im\psi_{n+1}^{(\alpha,\beta)}(\phi)\right)\right] A_{\alpha,\beta}(\phi)\, d\phi, \end{aligned}$$

where

$$I_n^{(\alpha,\beta)} := \frac{\Gamma(\alpha+n+2)\Gamma(\rho+n+1)}{2^{2\rho-1}\left(\Gamma(\alpha+1)\right)^2 (2n+\rho+1) n!\, \Gamma(\beta+n+1)}.$$

For all $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{\pm\theta\}$, we put

$$g_\theta(\phi) := \frac{f(\theta) - f(\phi)}{\cos(2\theta) - \cos(2\phi)}.$$

Since we have supposed that f is a piecewise continuous function on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then g_θ is also piecewise continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{\pm\theta\}$.
Furthermore, we have

$$\lim_{\phi\to\theta} g_\theta(\phi) = -\frac{1}{2}\frac{1}{\sin(2\theta)} f'(\theta).$$

And from hypothese i) of our theorem, we deduce that

$$\lim_{\phi\to-\theta} g_\theta(\phi) = \frac{1}{2}\frac{1}{\sin(2\theta)} f'(-\theta).$$

Under the assumption ii) of the theorem, these limits exist and are finite.
We still call g_θ the extension of g_θ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus, $g_\theta \in L^2_{\alpha,\beta}$.

In the following, we denote by

$$\begin{aligned}
\overset{\vee}{g}_\theta(\phi) &:= g_\theta(-\phi), \quad \phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\
g_\theta^1 &:= (g_\theta)_{|_{\left[0, \frac{\pi}{2}\right]}}, \\
g_\theta^2 &:= (g_\theta)_{|\left[-\frac{\pi}{2}, 0\right]}, \\
\overset{\vee}{\widehat{g_\theta^2}}(\phi) &:= g_\theta^2(-\phi), \quad \phi \in \left[0, \frac{\pi}{2}\right].
\end{aligned}$$

Now, we write

$$f(\theta) - S_n^f(\theta) = I_1 + I_2 + I_3 + I_4, \quad \text{where}$$

$$\begin{aligned}
I_1 &:= l_n^{(\alpha,\beta)} \varphi_{n+1}^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_n^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi)\, d\phi, \\
I_2 &:= -l_n^{(\alpha,\beta)} \varphi_n^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \varphi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi)\, d\phi, \\
I_3 &:= l_n^{(\alpha,\beta)} \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \Im\psi_{n+1}^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \Im\psi_n^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi)\, d\phi, \\
I_4 &:= -l_n^{(\alpha,\beta)} \frac{\lambda_n^{(\alpha,\beta)} \lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} \Im\psi_n^{(\alpha,\beta)}(\theta) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi) \Im\psi_{n+1}^{(\alpha,\beta)}(\phi) A_{\alpha,\beta}(\phi)\, d\phi.
\end{aligned}$$

Combining the fact that

$$l_n^{(\alpha,\beta)} \underset{+\infty}{\sim} \frac{1}{2^{2\rho}\left(\Gamma(\alpha+1)\right)^2} n^{2\alpha+1},$$

and the result (1), we get

$$l_n^{(\alpha,\beta)} \varphi_{n+1}^{(\alpha,\beta)}(\theta) \underset{+\infty}{\sim} n^{\alpha+\frac{1}{2}} \frac{\cos\left[(2n+2+\rho)|\theta| - (2\alpha+1)\frac{\pi}{4}\right]}{\sqrt{\pi}\Gamma(\alpha+1)A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)}.$$

Moreover, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi)\varphi_n^{(\alpha,\beta)}(\phi)A_{\alpha,\beta}(\phi)\,d\phi = \mathcal{F}_{\alpha,\beta}\left(\overset{\vee}{\widehat{g_\theta^1 + g_\theta^2}}\right)(n).$$

And since we know that

$$\mathcal{F}_{\alpha,\beta}\left(\overset{\vee}{\widehat{g_\theta^1 + g_\theta^2}}\right)(n) = o\left(n^{-(\alpha+\frac{1}{2})}\right),$$

then, $\lim\limits_{n\to+\infty} I_1 = 0$.

We use the same proof as for I_1 to show that

$$\lim_{n\to+\infty} I_2 = 0.$$

Otherwise, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g_\theta(\phi)\Im\psi_n^{(\alpha,\beta)}(\phi)A_{\alpha,\beta}(\phi)\,d\phi = \frac{1}{2i}\mathcal{F}\left(\overset{\vee}{g_\theta} - g_\theta\right)(n).$$

We have

$$\mathcal{F}\left(\overset{\vee}{g_\theta} - g_\theta\right)(n) = o\left(n^{-(\alpha+\frac{1}{2})}\right).$$

Furthermore, we know by (2) that

$$\Im\psi_{n+1}^{(\alpha,\beta)}(|\theta|) \underset{+\infty}{\sim} \frac{2^{2\rho}\Gamma(\alpha+1)}{\sqrt{\pi}}\frac{n^{-\left(\alpha+\frac{1}{2}\right)}}{A_{\frac{2\alpha-1}{4},\frac{2\beta-1}{4}}(\theta)}\sin\left[(2n+2+\rho)|\theta| - (2\alpha+1)\frac{\pi}{4}\right].$$

And since we have

$$\lim_{n\to+\infty}\frac{\lambda_n^{(\alpha,\beta)}\lambda_{n+1}^{(\alpha,\beta)}}{4(n+1)(n+\rho)} = 1,$$

then, we get

$$\lim_{n\to+\infty} I_3 = 0.$$

We use the same reasons as for I_3 to show that

$$\lim_{n \to +\infty} I_4 = 0.$$

Hence, we obtain

$$\lim_{n \to +\infty} \left[f(\theta) - S_n^f(\theta) \right] = \lim_{n \to +\infty} (I_1 + I_2 + I_3 + I_4) = 0.$$

Which achieves the proof.

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Thank you for your attention