Smoothing effect and Strichartz estimates for some time-degenerate Schrödinger equations

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Smoothing and Strichartz estimates

Smoothing estimates describe a typical property of dispersive equations consisting in a gain of smoothness of the solution of the IVP with respect to the smoothness of the initial datum and/or of the inhomogeneous term of the equation.

Strichartz estimates describe a gain of integrability of the solution of the IVP with respect to the integrability properties of the initial datum and/or of the inhomogeneous term of the equation.

These estimates are crucial to prove well-posedness results when dealing with nonlinear (smoothing estimates) and semilinear (Strichartz estmates) initial value problems.

Time-degenerate Schrödinger operators

In the sequel we will study the following classes of time-degenerate Schrödinger operators

- $\mathcal{L}_{\alpha,c} = i\partial_t + t^\alpha \Delta_x + c(t,x) \cdot \nabla_x, \quad \alpha > 0.$
  - Local smoothing effect.
  - Local well-posedness of the associated nonlinear IVP.
- $\mathcal{L}_b := i\partial_t + b'(t)\Delta_x$.
  - Strichartz estimates.
  - Local well-posedness of the associated semilinear IVP.

Smoothing for $\mathcal{L}_{\alpha,c} = i\partial_t + t^\alpha \Delta_x + c(t,x) \cdot \nabla_x$

Motivation

Schrödinger eq.	Smoothing effect	
Constant coefficients	Homogeneous: ✓ Inhomogeneous: ✓	Kato, Sjölin, Kenig-Ponce-Vega, Caustantin-Saut, Linares-Ponce, Kleinerman and many others..
Space-variable coefficients (elliptic case)	Homogeneous: ✓ Inhomogeneous: ✓	Doi, Kenig-Ponce-Vega-Rolvung
Non-degenerate time-variable coefficients	Homogeneous: ✓ Inhomogeneous: X	Sugimoto-Ruzhansky
Degenerate time-variable coefficients	Homogeneous: X Inhomogeneous: X	

Time-degenerate case: known results

Cicognani-Reissig: Local well-posedness of the *homogeneous* IVP for operators of the form $\mathcal{L}_{\alpha,c}$ both in Sobolev and Gevrey spaces.

Remark

No smoothing and Strichartz estimates are needed here, since the problem under consideration is *linear*.

Time-degenerate Schrödinger operators

We studied the operator $\mathcal{L}_{\alpha,c}$ in the following cases:

- $c \equiv 0$. By Fourier analysis methods we derived the weighted smoothing effect, both homogeneous and inhomogeneous, with a gain of 1/2 derivative;
- $c \not\equiv 0$. We used the pseudo-differential calculus to get a weighted homogeneous smoothing effect with a gain of 1/2 derivative and a weighted inhomogeneous smoothing effect with a gain of 1 derivative;

Remark. (case $c \equiv 0$) $\subset$ (case $c \not\equiv 0$)

The case $c \equiv 0$: $\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x, \quad \alpha > 0$

Let us consider the IVP

$$\begin{cases} \mathcal{L}_\alpha u = f \\ u(s,x) = u_s(x), \quad s \geq 0. \end{cases}$$

By using Fourier analysis methods and Duhamel's principle we get that the function u solving the problem above is

$$u(t,x) = W_\alpha(t,s)u_s(x) + \int_0^t W_\alpha(t',s)f(t',x)dt' = u_{\text{hom}} + u_{\text{inhom}},$$

where the **solution operator** $W_\alpha(t,s)$, $t,s \geq 0$, is given by

$$W_\alpha(t,s)u_s(x) := e^{i\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}\Delta}u_s := \int_{\mathbb{R}^n} e^{-i\left(\frac{t^{\alpha+1}-s^{\alpha+1}}{\alpha+1}|\xi|^2 - x\cdot\xi\right)}\widehat{u_s}(\xi)d\xi.$$

Notation: $W_\alpha(t,0) =: W_\alpha(t)$.

The case $c \equiv 0$: $\mathcal{L}_\alpha = i\partial_t + t^\alpha \Delta_x, \quad \alpha > 0$

Observe that the solution operator satisfies the following properties.

(i) $W_\alpha(t,t) = I$;

(ii) $W_\alpha(t,s) = W_\alpha(t,r)W_\alpha(r,s)$ for every $s,t,r \in [0,T]$;

(iii) $W_\alpha(t,s)\Delta_x u = \Delta_x W_\alpha(t,s)u$.

Moreover it is easy to see that

$$\|W_\alpha(t,s)u_s\|_{H^s_x} = \|u_s\|_{H^s_x}.$$

Notice that for $\alpha = 0$ we have the standard Schrödinger group.

Smoothing effect when $c \equiv 0$

By exploiting the properties of $W_\alpha(t,s)$ we derived the following smoothing effect for $\mathcal{L}_\alpha$.

Theorem. (F.-Staffilani) Let $W_\alpha(t) := W_\alpha(t,0)$, with $\alpha > 0$. Then, if $n = 1$, for all $\varphi \in L^2(\mathbb{R})$,

$$\sup_x \| t^{\alpha/2} D_x^{1/2} W_\alpha(t)\varphi\|^2_{L^2_t([0,T])} \lesssim \|\varphi\|^2_{L^2(\mathbb{R})}; \tag{1}$$

If $n \geq 2$, on denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ the family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then for all $\varphi \in L^2_x(\mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \int_0^T |t^{\alpha/2} D_x^{1/2} W_\alpha(t)\varphi(x)|^2 dt\, dx \right)^{1/2} \lesssim \|\varphi\|_{L^2(\mathbb{R}^n)}, \tag{2}$$

where $D_x^\gamma \varphi(x) = (|\xi|^\gamma \widehat{\varphi}(\xi))^\vee(x)$.

Theorem. (F.-Staffilani) Let $g \in L^1_t L^2_x([0,T]\times\mathbb{R}^n)$. Then, if $n=1$,

$$\|t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau\|_{L^\infty_x(\mathbb{R}) L^2_t([0,T])} \lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R})}. \tag{3}$$

If $n \geq 2$, denoting by $\{Q_\beta\}_{\beta\in\mathbb{Z}^n}$ a family of non overlapping cubes of unit size such that $\mathbb{R}^n = \bigcup_{\beta\in\mathbb{Z}^n} Q_\beta$, then, for all $g \in L^1_t L^2_x([0,T]\times\mathbb{R}^n)$,

$$\sup_{\beta\in\mathbb{Z}^n} \left(\int_{Q_\beta} \left\| t^{\alpha/2} D_x^{1/2} \int_0^t W_\alpha(t,\tau) g(\tau) d\tau \right\|^2_{L^2_t([0,T])} dx \right)^{1/2} \tag{4}$$

$$\lesssim \|g\|_{L^1_t L^2_x([0,T]\times\mathbb{R}^n)}$$

Local well-posedness result

Theorem. (F.-Staffilani) Let $k \geq 1$, then the IVP

$$\begin{cases} \mathcal{L}_\alpha u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases}$$

is locally well-posed in H^s for $s > n/2$ and its solution satisfies smoothing estimates.

Remark. The local well-posedness of the Cauchy problem above but with derivative nonlinearities follows from the equivalent result given in the general case $c \not\equiv 0$, which, as we already said, is true even in the particular case $c \equiv 0$.

Strategy of the proof: contraction argument

1. Consider the metric space

$$X = \{u : [0,T] \times \mathbb{R} \to \mathbb{C}; \|t^{\alpha/2} D_x^{1/2+s} u\|_{L_x^\infty L_t^2([0,T])} < \infty, \|u\|_{L_t^\infty([0,T])H_x^s} < \infty\},$$

equipped with the distance

$$d(u,v) = \|t^{\alpha/2} D_x^{1/2+s}(u-v)\|_{L_x^\infty L_t^2([0,T])} + \|u-v\|_{L_t^\infty([0,T])\dot{H}_x^s} + \|u-v\|_{L_t^\infty([0,T])L_x^2}.$$

2. Consider the map

$$\Phi : X \to X, \quad \Phi(u) = W_\alpha(t)u_0 + \int_0^t W_\alpha(t,\tau)u|u|^{2k}(\tau)d\tau.$$

3. Prove that Φ is a contraction on $B \subset X$ using the smoothing estimates. Finally apply fixed point theorem to get the result.

The case $c \not\equiv 0$. Local weighted smoothing effect

We considered the IVP

$$\begin{cases} \partial_t u = it^\alpha \Delta_x u + ic(t,x) \cdot \nabla_x u + f(t,x) \\ u(0,x) = u_0(x). \end{cases} \tag{5}$$

Theorem. (F.-Staffilani)
Let $u_0 \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$. Assume that, for all $j = 1, ..., n$, c_j is such that $c_j \in C([0,T], C_b^\infty(\mathbb{R}^n))$ and there exists $\sigma > 1$ such that

$$|\mathrm{Im}\, \partial_x^\gamma c_j(t,x)|, |\mathrm{Re}\, \partial_x^\gamma c_j(t,x)| \lesssim t^\alpha \langle x \rangle^{-\sigma-|\gamma|}, \quad x \in \mathbb{R}^n, \tag{6}$$

and denote by $\lambda(|x|) := \langle x \rangle^{-\sigma}$.
Let also Λ^s be the Fourier multiplier $\widehat{\Lambda^s u}(\xi) = \langle \xi \rangle^s \widehat{u}(\xi)$. Then

(i) If $f \in L^1([0,T]; H^s(\mathbb{R}^n))$ then the IVP (5) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\le t\le T} \|u(t)\|_s \le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s + \int_0^T \|f(t)\|_s dt \right);$$

(ii) If $f \in L^2([0,T]; H^s(\mathbb{R}^n))$ then the IVP (5) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist two positive constants C_1, C_2 such that

$$\sup_{0\le t\le T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2} u\right|^2 \lambda(|x|)dx\, dt$$

$$\le C_1 e^{C_2(\frac{T^{\alpha+1}}{\alpha+1}+T)} \left(\|u_0\|_s^2 + \int_0^T \|f(t)\|_s^2 dt \right);$$

(iii) If $\Lambda^{s-1/2} f \in L^2([0,T]\times\mathbb{R}^n; t^{-\alpha}\lambda(|x|)^{-1} dtdx)$ then the IVP (5) has a unique solution $u \in C([0,T]; H^s(\mathbb{R}^n))$ and there exist positive constants C_1, C_2 such that

$$\sup_{0\le t\le T} \|u(t)\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^\alpha \left|\Lambda^{s+1/2} u\right|^2 \lambda(|x|) dx\, dt$$

$$\le C_1 e^{C_2 \frac{T^{\alpha+1}}{\alpha+1}} \left(\|u_0\|_s^2 + \int_0^T \int_{\mathbb{R}^n} t^{-\alpha} \lambda(|x|)^{-1} \left|\Lambda^{s-1/2} f\right|^2 dx\, dt \right).$$

Remarks on the conditions

We remark that it is natural to require the previous conditions on the term c in order to have the l.w.p. of the IVP. Indeed, even in the nondegenerate case (with $c = c(x)$), decay conditions on $\mathrm{Re}\, c$ are necessary for the local well-posedness of the linear Cauchy problem to hold.
The main problems when proving the smoothing effect for $\mathcal{L}_{\alpha,c}$ are given by the presence of the time degeneracy t^α in the second order term and by the presence of the first order term $c \cdot \nabla_x$. Due to these considerations it is clear that conditions on $c(t,x)$ are necessary to control the behavior of the operator, and, specifically, conditions relating the coefficient t^α and the coefficients $c_j(t,x)$. Our strategy aims to control gradient term in the energy estimate by exploiting the form of the second order term and the decay properties of c.

Strategy of the proof

- We use a pseudo-differential weight (given by Doi's Lemma) to define a norm N equivalent to the H^s-norm $\|\cdot\|_s$.
- We perform an energy estimate in terms of the norm N.
- We use the properies of the pseudo-differential weight to apply the sharp Gårding inequality to absorb the (dangerous) lower order terms.
- We obtain the desired inequalities from which we get existence and uniqness of the solution by means of standard functional analysis arguments.

Local well-posedness of the NLIVP

We then considered the NLIVP

$$\begin{cases} \mathcal{L}_{\alpha,c}u = \pm u|u|^{2k} \\ u(0,x) = u_0(x), \end{cases} \tag{7}$$

and

$$\begin{cases} \mathcal{L}_{\alpha,c}u = \pm t^\beta \nabla u \cdot u^{2k}, \quad \beta \geq \alpha > 0, \\ u(0,x) = u_0(x). \end{cases} \tag{8}$$

Theorem. (F.-Staffilani) Let $\mathcal{L}_{\alpha,c}$ be such that condition (6) is satisfied. Then the IVP (7) is locally well posed in H^s for $s > n/2$.

Theorem. (F.-Staffilani) Let $\mathcal{L}_{\alpha,c}$ be such that condition (6) is satisfied with $\sigma = 2N$ (thus $\lambda(|x|) = \langle x\rangle^{-2N}$) for some $N \geq 1$, and $s > n + 4N + 3$ such that $s - 1/2 \in 2\mathbb{N}$. Let $H^s_\lambda := \{u_0 \in H^s(\mathbb{R}^n); \lambda(|x|)u_0 \in H^s(\mathbb{R}^n)\}$, then the IVP (8) with $\beta \geq \alpha > 0$, is locally well posed in H^s_λ.

The class $\mathcal{L}_b = i\partial_t + b'(t)\Delta_x$

We now consider operators of the form

$$\mathcal{L}_b := \partial_t + ib'(t)\Delta_x,$$

where $b \in C^1(\mathbb{R})$ is such that $b'(0) = 0$ and b has either finitely or infinitely many critical points.

Example 1

$$\mathcal{L}_{b/\alpha,c} = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1}/\alpha,c} = \partial_t + it^\alpha\Delta + c(t,x)\cdot\nabla_x, \quad \alpha > 0;$$

Example 2

$$\mathcal{L}_b = \mathcal{L}_{e^t - t - 1} = \partial_t + i(e^t - 1)\Delta;$$

Example 3

$$\mathcal{L}_b = \mathcal{L}_{\cos(t)} := \partial_t u - i\sin(t)\Delta.$$

The class $\mathcal{L}_b = i\partial_t + b'(t)\Delta_x$

Motivation

Schrödinger equation with ...	Strichartz estimates	
Constant coefficients	✓	Strichartz, Kenig-Ponce-Vega, Linares-Ponce, Ginibre-Velo, Keel-Tao, Walther and others..
Space-variable coefficients (elliptic case)	✓	Staffilani-Tataru, Marzuola, Metcalfe-Tataru and others...
Degenerate space-variable coefficients	✓	Salort
Time-variable coefficients	X	

Strichartz estimates for $\mathcal{L}_b$

Recall that

$$\mathcal{L}_b := \partial_t + ib'(t)\Delta_x,$$

with $b \in C^1(\mathbb{R})$ and such that $b'(0) = 0$ and b has either finitely or infinitely many critical points. Due to the form of $\mathcal{L}_b$ we have that the solution at time t of the IVP

$$\begin{cases} \mathcal{L}_b u(t,x) = g(t,x), \\ u(s,x) = u_s(x) \end{cases} \tag{9}$$

can be written as

$$u(t,x) = e^{i(b(t)-b(s))\Delta} u_s(x) + \int_s^t e^{i(b(\tau)-b(s))\Delta} g(\tau)d\tau = u_{\text{hom}} + u_{\text{inhom}},$$

$$e^{i(b(t)-b(s))\Delta} u_s(x) := \int_{\mathbb{R}^n} e^{ix\cdot\xi + i(b(t)-b(s))|\xi|^2} \widehat{u}_s(\xi)d\xi.$$

By using the previous formula one can prove the following local in time Strichartz estimates.

Weighted local Strichartz estimates for $\mathcal{L}_b$

Notations. We shall denote by $L^q_t L^p_x := L^q_t(\mathbb{R}; L^p_x(\mathbb{R}^n))$, and, when not confusing, we shall use the same notation $L^q_t L^p_x := L^q_t([0,T]; L^p_x(\mathbb{R}^n))$ when the time interval is finite.

Definition. (*Admissible pairs*) Given $n \geq 1$ we shall call a pair of exponents (q,p) admissible if $2 \leq q, p \leq \infty$, and

$$\frac{2}{q} + \frac{n}{p} = \frac{n}{2}, \quad \text{with} \quad (q,p,n) \neq (2,\infty,2).$$

Theorem. (F.-Ruzhansky) Let $b \in C^1([0,T])$ be such that, for any fixed T, $\sharp\{t \in [0,T]; b'(t) = 0\} = k \geq 1$ and $b'(0) = 0$. Then, for any (q,p) admissible pair such that $2 < q, p < \infty$, the following inequalities hold:

the weighted homogeneous Strichartz estimate

$$\| |b'(t)|^{1/q} e^{ib(t)\Delta}\varphi \|_{L^q_t L^p_x} \leq C(n,q,p,k)\|\varphi\|_{L^2_x(\mathbb{R}^n)}, \qquad (10)$$

where

$$\|e^{ib(t)\Delta}\varphi\|_{L^\infty_t L^2_x} \leq \|\varphi\|_{L^2_x(\mathbb{R}^n)};$$

the weighted inhomogeneous Strichartz estimate

$$\| |b'(t)|^{1/q} \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s)ds \|_{L^q_t L^p_x} \leq C(n,q,p,k)\| |b'|^{1/q'} g\|_{L^{q'}_t L^{p'}_x}; \qquad (11)$$

where

$$\| \int_0^t |b'(s)| e^{i(b(t)-b(s))\Delta} g(s)ds \|_{L^\infty_t L^2_x} \leq C(n,q,p,k)\| |b'|^{1/q'} g\|_{L^{q'}_t L^{p'}_x}.$$

Local well-posedness of the semilinear problem

Let us consider the semilinear IVP

$$\begin{cases} \partial_t u + ib'(t)\Delta u = \mu|b'(t)||u|^{p-1}u, & \mu\in\mathbb{R} \\ u(0,x) = u_0(x), \end{cases} \tag{12}$$

then the following local well-posedness result holds.
Theorem. (F.-Ruzhansky) Let $1 < p < \frac{4}{n} + 1$ and $b \in C^1([0,+\infty))$ be such that $\sharp\{t\in[0,\tilde{T}]; b'(t) = 0\}$ is finite for any $\tilde{T} < \infty$ and $b'(0) = 0$. Then for all $u_0 \in L^2(\mathbb{R}^n)$ there exists $T = T(\|u_0\|_2, n, \mu, p) > 0$ such that there exists a unique solution u of the IVP (12) in the time interval $[0,T]$ with

$$u \in C([0,T]; L^2(\mathbb{R}^n)) \bigcap L^q_t([0,T]; L^{p+1}_x(\mathbb{R}^n))$$

and $q = \frac{4(p+1)}{n(p-1)}$. Moreover the map $u_0 \mapsto u(\cdot,t)$, locally defined from $L^2(\mathbb{R}^n)$ to $C([0,T); L^2(\mathbb{R}^n))$, is continuous.

Application

By the previous theorem we can conclude the local well-posedness for the semilinear IVP associated with:

Example 1

$$\mathcal{L}_{b/\alpha,0} = \mathcal{L}_{\frac{t^{\alpha+1}}{\alpha+1},\Delta} = \partial_t + it^\alpha\Delta, \quad \alpha \geq 1$$

$\sharp\{t \in [0,T]; b'(t) = 0\} = 1$ for any $0 < T < \infty$.

Example 2

$$\mathcal{L}_b = \mathcal{L}_{e^t - t - 1} = \partial_t + i(e^t - 1)\Delta,$$

$\sharp\{t \in [0,T]; b'(t) = 0\} = 1$ for any $0 < T < \infty$.

Example 3

$$\mathcal{L}_b = \mathcal{L}_{\cos(t)} := \partial_t u - i\sin(t)\Delta$$

$\sharp\{t \in [0,T]; b'(t) = 0\} = k \geq 1$ for any $0 < T < \infty$.

Remarks

Together with G. Staffilani we have recently considered operators of the form $\mathcal{L}_b$ on $\mathbb{T}^d$, with $d = 2, 1$. We obtained sharp weighted (in time) Strichartz estimates and proved local well-posedness results for the cubic (resp. quintic) semilinear IVP (where the nonlinearity also contains a time-dependent function).

Additionally, still in the toroidal setting, we considered some particular (nondegenerate) space-variable coeffficients Schrödinger operators. In this case we proved local well-posedness results for the cubic (resp. quintic) semilinear IVP by means of the standard sharp Stichartz estimates involving the so-called Bourgain spaces $X^{s,b}(\mathbb{R} \times \mathbb{T}^d)$.

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