Lipschitz Conditions in Damek-Ricci Spaces.

Radouan DAHER

Faculty of Sciences Aïn Chock, Hassan II University of Casablanca-Morocco.

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Introduction

- The studies of the convergence and the rate of decay of Fourier transform/coefficients are among the most classical problems in Fourier analysis.
- Lipschitz condition states that :

$$\begin{cases} 0 < \alpha \leqslant 1 \\ |f(x) - f(x')| \leqslant M|x - x'|^{\alpha}; \end{cases} \tag{1}$$

- It was first considered by Lipschitz in 1864 while studying the convergence of the Fourier series of a periodic function.
- He proved that the inequality (1) is sufficient to have that the Fourier series of f converges everywhere to the value of f.

Introduction

- An other criterion was introduced by Dini in 1872 whose conclusion states that the convergence is in addition uniform.
- Starting from the Riemann-Lebesgue theorem relating the integrability of a function on the torus $\mathbb{T}^n$ and the convergence of its Fourier coefficients, through the Hausdorff-Young inequality relating the integrability of a function and its Fourier transform.
- In this vein, Titchmarsh (1937) showed that the decay of Fourier transform can be improved for univariate functions.

satisfying a Lipschitz condition defined by smoothness.
His result reads as follows:

Theorem 1 ([T] ,th 84)

If f belongs to the Lipschitz class $\mathrm{Lip}(\eta, p)$ *in the* $\mathbb{L}^p$ *norm on the real line that is :*

$$\omega_p(f,t) = \|f(\cdot + t) - f(\cdot)\|_{\mathbb{L}^p(\mathbb{R})} = O(|t|^\alpha) \qquad as \qquad t \to 0,$$

then its Fourier transform $\widehat{f}$ *belongs to* $\mathbb{L}^\delta(\mathbb{R})$ *for*

$$\frac{p}{p+\alpha p-1} \leqslant \delta \leqslant \frac{p}{p-1}; \quad \begin{array}{l} 0<\alpha\leqslant 1 \\ 1<p\leqslant 2 \end{array}$$

He also proved ([T]; th 85) another reversible form in the $\mathbb{L}^2$ case, namely:

Theorem 2

Let $0 < \alpha \leqslant 1$ *and* $f \in \mathbb{L}^2(\mathbb{R})$. *Then*

$$f \in \mathrm{Lip}(\alpha, 2) \quad \textit{if and only if} \quad \int_{|\lambda| \geqslant r} |\hat{f}(\lambda)|^2 d\lambda = O\left(r^{2\alpha}\right); r \to \infty$$

- In 1960, Krovokin considered Dini-Lipschitz class as the set of functions in $\mathbb{L}^p(\mathbb{R})$ such that :

$$\|f(\cdot + h) - f(\cdot)\|_{\mathbb{L}^p(\mathbb{R})} = o\left(ln\left(\frac{1}{h}\right)^{-1}\right).$$

- An extension of these theorems to function of several variables on $\mathbb{R}^n$ and on the torus group $\mathbb{T}^n$ was studied by Younis $(1970, 1974)$.
- Younis in 1986 showed that the result of Titchmarsh's theorem ([T], th 84) does not hold for the Dini-Lipschitz functions:

It does not improve the Hausdorff-Young inequality and the conclusion is that $\widehat{f}$ belong to $\mathbb{L}^{p'}(\mathbb{R})$ (where $\frac{1}{p}+\frac{1}{p'}=1$). Therefore, he considered some conditions which are rather situated in between the Lipschitz and the Dini-Lipschitz conditions. These were inspired from Weiss and Zygmund. It states that:

$$\|f(\cdot+h)-f(\cdot)\|_{\mathbb{L}^p(\mathbb{R})}=O\left(h^{\alpha}\ln\left(\frac{1}{h}\right)^{-\delta}\right); \qquad \delta\geqslant 0.$$

- Known results:
  - $\mathbb{R}^n$, Younis (1970).
  - π^n, Younis (1974)
  - $\mathbb{R}^n$, Daher-El Hamma (2014)
    $*$ Here in this paper we considered in $\mathbb{L}^2(\mathbb{R}^n)$ the spherical mean operator :

$$M_h f(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + h\omega) d\omega.$$

$$\widehat{M_h f}(\xi) = j_{\frac{n-2}{2}}(h|\xi|)\hat{f}(\xi)$$

  - Later, analogous results were given, where considering generalized Fourier transform:
  - Bessel Hypergoup (Daher-El Hamma 2016).
  - Dunkl Setting (Daher-El Hamma 2015).
  - Jacobi-Dunkl (Daher-El ouadih 2016).
  - Laguerre Hypergoup (S.Negazaoui 2017).
  - Discrete Fourier-Jacobi (Daher-EL ouadih 2017).

- On certain Lie group (Younis 2000)
- On $SU(2)$ (Jacobi polynomials)
- On $SU(1,1)$ (Jacobi functions)
- On $SL(2,\mathbb{R})$ (Hypergeometric functions)
- $G = KAN \cdot$ (Iwasawa decomposition)
- Natural question prises as to the capability of carrying the Known analysis on $G = KAN$.

Younis conjecture : The answer will be affirmative.

- Younis support this by emphasizing that on the subgroup A the problem boils, down to order of magnitude of the Melin transform of Lipschitz function ([Y]; 1997).
- On the subgroup N the question reduces to the Radon transform of Certain Lipschitz class.

- Younis (1998) has extended Titchmarsh's theorem on the Hyperbolic plane H^2.
- Platonov (2005) has extended Younis 's result on H^2 to N.C.S.S of rank one.
- In 2016 Daher and El ouadih has characterized the Deni-Lipschitz functions for the Helgason Fourier transform on rank one symmetric spaces.

- D-Delgado.Ruzhansky (2019) Compact homogeneous manifold.
- ...etc.
- In this talk, we investigate among other things the validity of classical Titchmarsh theorems in the case of functions of the wider Lipschitz and Dini-Lipschitz class in the context of Damek-Ricci spaces, also known as harmonic NA groups This generalizes the corresponding result for N.C.S.S of rank one.

- Our current interest in this theme stems from a result of Kumar and al.(2010) which is based on the work of Bray and Pinsky (2008).

Essentials about Damek-Ricci Spaces

- For basis of Damek-Ricci Spaces and harmonic analysis on them, we can refer to seminal research papers. In cimpa school organized in Morocco (1999) Pr.F.Rouvière gave a very nice lecture : (in French) Espaces de Damek-Ricci géometrie et analyse (Semin. Congres;7, soc . Math. France, Paris 2003).

Essentials about Damek-Ricci Spaces

- A Damek-Ricci Space is one-dimensional extension of a generalized Heisenberg group and a Lie group with the Lie algebra of Iwasawa type.
  It is solvable Lie group with a left a left invariant metric and is a Riemannian manifold which includes all rank one symmetric spaces of the non compact type, except from these, Damek-Ricci Spaces one harmonic manifold in general non symmetric.

Essentials about Damek-Ricci Spaces

- One of the interesting features of thes spaces is that the radial analysis on these spaces behaves similar to the hyperbolic spaces as observed(Anker and al 1996) and therefore it fits into the perfect setting of Jacobi analysis developed by Flensted-Jensen and Koorwinder (1979).

- Let $S = NA$ be a semi direct product of N and A under certain action.
- For a measurable function f on S, the Fourier-Helgason transform is defined as :

$$\widehat{f}(\lambda, n) = \int_S f(x) P_\lambda(x, n) dx$$

whenever the integral converge. and $P_\lambda(x, n)$ is an eigenfunction a Laplace-Beltrami operator Δ_S on S with eigenvalue $-\left(\lambda^2 + \frac{\theta^2}{4}\right)$
$P_\lambda(x, n)$ is certain complex power of Poisson kernel.

$*$ It is known that for $f \in C_c^{\infty}(S)$ the following Fourier inversion and the Plancherel formula holds (see Astengo and al 1997).
$*$ For $f \in C_c^{\infty}(S)$; $\forall x \in S$

$$f(x) = c \int_{\mathbb{R}} \int_{\mathbb{N}} \widehat{f}(\lambda, n) P_{\lambda}(x, n).|c(\lambda)|^{-2} d\lambda dn$$

$*$ The Fourier transform extends from $C_c^{\infty}(S)$ to an isomerty from $\mathbb{L}^2(S)$ into the space $\mathbb{L}^2(\mathbb{R}^+ \times \mathbb{N}; |c(\lambda)|^{-2} d\lambda dn)$

The elementary spherical functions $\Phi_\lambda(x)$ is defined by

$$\Phi_\lambda(x) = \int_{\mathbb{N}} P_\lambda(x,n)P_{-\lambda}(x,n)dn.$$

. As consequence of the relation between Jacobi function $\Phi_\lambda^{\alpha,\beta}$ $(\alpha > \beta > -\frac{1}{2})$ and elementary spherical functions ϕ_λ as

$$\phi_\lambda(t) = \Phi_{2\lambda}^{\alpha,\beta}(t/2)$$

we have these crucial lemmas

Lemma 3 ([P] 1999)

(i) $|\phi_\lambda(t)| \leq 1$

(ii) $|1-\phi_\lambda(t)| \leq \frac{t^2}{2}\left(\lambda^2 + \frac{\theta^2}{4}\right)$

(ii) *There exists a constant $c > 0$ depending only on λ, such that*

$$|1-\phi_\lambda(t)| \geq c \qquad \textit{for} \qquad \lambda t \geq 1$$

Lemma 4 ([B-P] 2012)

Let $\alpha > -\frac{1}{2}$, $\quad \frac{1}{2} \leq \beta \leq \alpha$ *and let* $0 < \gamma_0 < \rho$ *there exist a positive constant* $c_1 = c(\alpha, \beta, \rho)$ *such that*

$$|1 - \Phi_{\lambda+2\gamma}(t)| \geq c_1 min\left\{1; (\lambda t)^2\right\}$$

for all $|\gamma| \leq \gamma_0, \quad \lambda \in \mathbb{R}$ *and* $t > 0$.

- Let σ_t be the normalized surface measure of the geodesic sphere of radius t. Then π_t is a non negative radial measure.
- The spherical mean operator M_t on a suitable function space on S is defined by

$$M_t f = f * \sigma_t$$

- It can proves that

$$M_t f(x) = \mathcal{R}\left(f^x\right)(t)$$

where f^x denotes the right translation of function f by x, and $\mathcal{R}$ is the radialization of operator defined, for suitable function f by:

$$\mathcal{R}f(s) = \int_{S_\nu} f(y) d\sigma_\nu(y)$$

where $\nu = r(x) = \mu(c(x), 0)$
here c is the Cayley transform and $d\nu_\nu$ is the normalized surface induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S : \mu(y, e) = \nu\}$

- It is easy to see that $\mathcal{R}f$ is a radial function and for any radial function f

$$\mathcal{R}f = f.$$

Kumar and el. proved the following inequality.
For $1 < p \leq 2, p \leqslant q \leqslant p' = \frac{P}{P-1}$ and $f \in \mathbb{L}^{p'}(S)$ we have

$$\int_{\mathbb{R}} \min\left\{1;(\lambda t)^{2p'}\right\}\left(\int_N \widehat{f}\,(\lambda + i\gamma_p pn)^q\, dn\right)^{p'/q} d\mu(\lambda) \leqslant C_{p,q}^{p'} \left\|M_t f - f\right\|_p^{p'} \quad (2.2)$$

where $d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda$.

Consequently for a radial function f, $M_t f$ is the usual translation of f by t.

- In [Kumar-S.S , 2010] the authors proved that, for a suitable function f on S,

$$\tilde{M}_t f\,(\lambda, n) = \phi_\lambda(a_t)\tilde{f}\,(\lambda, n)$$

  Whenever both make sense.
  Also $M_t f \longrightarrow f$ as $t \to 0$
- We have also : $\|\mathrm{M}_t f\|_2 \leqslant \phi_0(t)\| f\|_2$

In this section, we give the main result of this talk But first we need to define the Lipschitz class

Definition 5

Let $0 < \eta \leqslant 1$ A function $f \in \mathbb{L}^p(S)$ is said to be in the Damek-Ricci-Lipschitz class, denoted by $Lip(\eta, p)$ if it satisfies $\|M_t f - f\|_p = O\left(|t|^\eta\right); t \to 0$

The following Theorem represents a quantified Riemann-Lebesgue lemma, item (1), and is an extension of results in one dimension given in Titchmarsh.

Theorem 6

Let $0 < p \leq 2$ *and* $p' = p/(p-1)$.

1. *If* $f \in Lip(\eta, p)$, $0 < \eta \leq 1$, *then*

$$\int_{|\lambda|\geq r} \int_N |\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)|^{p'} dn d\lambda = O\left(r^{-p'\eta - d + 1}\right), \quad as \quad r \to \infty;$$

2. *when* $p = 2$ *and* $0 < \eta < 1$, *the converse statement holds as well.*

Proof.

1. The proof of this result is immediate from the estimate (2.2).

2. For the converse when $p = 2$ the same proof presented in (Daher-El ouadih 2016. N.C.S.S of rank one)

For $f \in L^p(S)$, we define the finite differences of first and higher order as follows:

$$\begin{aligned} \Delta_t^1 f &= \Delta_t f = (I - M_t) f, \\ \Delta_t^k f &= \Delta_t(\Delta_t^{k-1} f) = (I - M_t)^k f, \quad k = 2, 3, ..., \end{aligned}$$

where I is the unit operator in the space $L^p(S)$.
Consequently, for each $f \in L^p(S)$,

$$\widetilde{\Delta_t^k f}(\lambda + i\gamma_{p'}\rho, n) = (1 - \phi_{\lambda + i\gamma_{p'}\rho}(a_t))^k \widetilde{f}(\lambda + i\gamma_{p'}\rho, n),$$

and, by Plancherel formula, we have (3.1)

$$\left\| \Delta_t^k f \right\|_2^2 = \int_0^{+\infty} \int_N |1 - \phi_{\lambda + i\gamma_{p'}\rho}(a_t)|^{2k} |\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)|^2 \, |c(\lambda)|^{-2} \, d\lambda dn,$$

By analogy with the proof of Theorem 6, we can establish from formula 3.1 the following result:

Theorem 7

Let $0 < p \leq 2$ *and* $p' = p/(p-1)$.

1. *If* $\left\|\Delta_t^k f\right\|_2 = O(|t|^\eta)$, $0 < \eta \leq 1$, *then*

$$\int_{|\lambda|\geq r} \int_N |\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)|^{p'} dnd\lambda = O\left(r^{-p'\eta - d + 1}\right), \quad as \quad r \to \infty;$$

2. *when* $p = 2$, $0 < \eta < 1$ *and* $k = 1, 2, \ldots$, *the converse statement holds as well.*

We now state our second main result which extends the integrability Theorem 1 to Damek-Ricci spaces.

Theorem 8

Let $1 < p \leq 2$, $p' = p/(p-1)$, $0 < \eta \leq 1$ and $f \in Lip(\eta, p)$. Then its transform $\widetilde{f}(. + i\gamma_{p'}\rho, .)$ is in $L^{\delta}(\mathbb{R} \times N)$ with respect to the Plancherel measure $dnd\mu(\lambda)$ for every δ,

$$\frac{pd}{d(p-1) + p\eta} < \delta \leq p'.$$

Proof. Using formula (2.2), Applying the Hölder's inequality with $\delta \leq p'$.

Definition 9

Let $0 < \eta \leq 1$ and $\gamma \geq 0$. A function $f \in L^p(S)$ is said to be in the Damek-Ricci-Dini-Lipschitz class, denoted by $DLip(\eta, \gamma, p)$, if

$$\|M_t f - f\|_p = O\left(|t|^\eta / (\log \frac{1}{|t|})^{-\gamma}\right) \quad \text{as} \quad |t| \to 0.$$

By using the same tricks of calculation that we have already used to show the previous theorems, we prove the following theorems.

Theorem 10

Let $0 < p \leq 2$ *and* $p' = p/(p-1)$.

1. *If* $f \in DLip(\eta, \gamma, p)$, $0 < \eta \leq 1$, $\gamma \geq 0$, *then*

$$\int_{|\lambda| \geq r} \int_N |\widetilde{f}(\lambda + i\gamma_{p'}\rho, n)|^{p'} dn d\lambda = O\left(r^{-p'\eta - d + 1} (\log r)^{-p'\gamma}\right), \quad as$$

2. *when* $p = 2$, $\gamma \geq 0$ *and* $0 < \eta < 1$, *the converse statement holds as well.*

Theorem 11

Let $1 < p \leq 2$, $p' = p/(p-1)$, $0 < \eta \leq 1$, $\gamma \geq 0$ *and* $f \in DLip(\eta, \gamma, p)$. *Then its transform* $\widetilde{f}(. + i\gamma_{p'}\rho, .)$ *is in* $L^{\delta}(\mathbb{R} \times N)$ *with respect to the Plancherel measure* $dnd\mu(\lambda)$ *for every* δ,

$$\frac{pd}{d(p-1) + p\eta} < \delta \leq p'.$$

We conclude with the following result:

Theorem 12

Let $\eta > 2$, $\gamma \geq 0$ and $f \in DLip(\eta, \gamma, 2)$, then $f = 0$ a.e.

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End

Thank you for your attention