Well-posedness of Hardy-Hénon parabolic equation

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August 3, 2021
ISAAC
Harmonic Analysis and Partial Differential Equations

Hardy-Hénon parabolic equation

We consider the following Cauchy problem:

$$\begin{cases} \partial_t u - \Delta u = |x|^\gamma |u|^{\alpha-1}u, & t>0,\ x\in\mathbb{R}^d, \\ u(0) = u_0 \in L^q_s(\mathbb{R}^d), \end{cases} \tag{HH}$$

where, $d \geq 1$, $\gamma > -\min\{2,d\}$ and $\alpha > 1 + \frac{2+\gamma}{d}$.

Properties of (HH) when $\gamma \neq 0$ No translation invariance. No classical solution when $\gamma < 0$.

Weighted Lebesgue space $s \in \mathbb{R}$, $1 \leq q \leq \infty$,

$$L^q_s(\mathbb{R}^d) := \left\{ f \in \mathcal{M}(\mathbb{R}^d)\,;\, \|f\|_{L^q_s} = \left(\int_{\mathbb{R}^d} (|x|^s |f(x)|)^q \, dx \right)^{\frac{1}{q}} < \infty \right\}$$

Property ($\partial_t u - \Delta u = |x|^\gamma |u|^{\alpha-1} u$)

Scaling transform (HH) is invariant under the transformation

$$u_\lambda(t,x) := \lambda^{\frac{2+\gamma}{\alpha-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Critical scale Under the scaling transform,

$$\|u_\lambda(0)\|_{L^q_s} = \lambda^{-s+\frac{2+\gamma}{\alpha-1}-\frac{d}{q}} \|u(0)\|_{L^q_s}$$

Let $s_c := \frac{2+\gamma}{\alpha-1} - \frac{d}{q}$. We say that $u_0 \in L^q_s(\mathbb{R}^d)$ is

- scale-subcritical when $s < s_c$, i.e., $-s + \frac{2+\gamma}{\alpha-1} - \frac{d}{q} > 0$.
- scale-critical when $s = s_c$, i.e., $-s + \frac{2+\gamma}{\alpha-1} - \frac{d}{q} = 0$.
- scale-supercritical when $s > s_c$, i.e., $-s + \frac{2+\gamma}{\alpha-1} - \frac{d}{q} < 0$.

Fujita exponent $\gamma > -\min\{2, d\}$

$$1 + \frac{2+\gamma}{d}$$

separates the generation of global solutions and finite-time blowup solutions (Qi 1998).

- $\alpha \leq 1 + \frac{2+\gamma}{d}$: For any initial data, if there exists a positive solution, then it blows up in finite time.
- $\alpha > 1 + \frac{2+\gamma}{d}$: For reasonably smooth data, there exists a global solution.

Aim

A unified local theory treating all cases of Hardy, Fujita and Hénon.
Sharp conditions for the well-posedness / ill-posedness of the problem.

Results

In what follows q is a suitable real number.

Theorem (C.-Ikeda-Taniguchi, arXiv:2104.14166)

Let $\gamma > -\min(2, d)$ and $\alpha > 1 + \frac{2+\gamma}{d}$.

(i) If $s \leq s_c = \frac{2+\gamma}{\alpha-1} - \frac{d}{q}$ (Scale-critical or subcritical), then (HH) is locally WP in $L^q_s(\mathbb{R}^d)$.

(ii) $s > s_c$ (Scale-supercritical), there exists some function u_0 in $L^q_s(\mathbb{R}^d)$ such that (HH) with initial data u_0 does not have a local solution.

Known results on the LWP

- Wang (1993): $\gamma \in \mathbb{R}$, α : Sobolev-super, LWP in $C_B(\mathbb{R}^d)$.
- Ben Slimene-Tayachi-Weissler (2017): $-2 < \gamma < 0$, LWP in $L^p(\mathbb{R}^d)$.
- Madjoub (2020): $\gamma \in \mathbb{R}$, LWP in $C_B(\mathbb{R}^d)$.
- Tayachi (2020): $-2 < \gamma < 0$, Unconditional uniqueness/non-uniqueness.

Auxiliary norm (Kato norm)

$$\|u\|_{\mathcal{K}^s(T)} := \sup_{0\le t\le T} t^{\frac{s_c-s}{2}} \|u\|_{L^q_s}, \quad s_c = \frac{2+\gamma}{\alpha-1} - \frac{d}{q}$$

Mild solution We call $u = u(t,x) \in C([0,T]; L^q_{\tilde{s}}(\mathbb{R}^d)) \cap \mathcal{K}^s(T)$ satisfying the following a mild solution.

$$u(t,x) = e^{t\Delta}u_0(x) + \int_0^t e^{(t-\tau)\Delta} \left\{ |\cdot|^\gamma |u(\tau,\cdot)|^{\alpha-1} u(\tau,\cdot) \right\}(x) d\tau.$$

$$T_m = T_m(u_0) := \sup \left\{ T > 0 \,; \begin{array}{l} \text{There exists a unique solution } u \text{ of (HH)} \\ \text{in } C([0,T]; L^q_{\tilde{s}}(\mathbb{R}^d)) \cap Y \text{ with initial data } u_0 \end{array} \right\}.$$

Critical LWP : $s = s_c$

LWP in $L^q_{s_c}(\mathbb{R}^d)$

Let $\gamma \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ be such that, $\gamma > -\min(2, d)$ and $\alpha > 1 + \frac{2+\gamma}{d}$. Let $q \in \mathbb{R}$ and $s \in \mathbb{R}$ be such that

$$0 < \frac{1}{q} < \min\left\{\frac{1}{\alpha}, \frac{2}{d(\alpha-1)}, \frac{1}{\alpha-1}\left(1 - \frac{2+\gamma}{d(\alpha-1)}\right)\right\},$$

$$\frac{s_c+\gamma}{\alpha} \leq s < \min\left\{\frac{d+\gamma}{\alpha} - \frac{d}{q}, s_c\right\}.$$

Then

(i) (Existence) For any $u_0 \in L^q_{s_c}(\mathbb{R}^d)$ with $q < \infty$ there exist a positive number T and an $L^q_{s_c}(\mathbb{R}^d)$-mild solution u to (HH) satisfying

$$\|u\|_{\mathcal{K}^s(T)} \leq 2\|e^{t\Delta}u_0\|_{\mathcal{K}^s(T)}.$$

LWP in $L^q_{s_c}(\mathbb{R}^d)$

(ii) (Uniqueness) Let $T > 0$. If $u, v \in \mathcal{K}^s(T)$ satisfy (HH) with $u(0) = v(0) = u_0 \in L^q_{s_c}(\mathbb{R}^d)$, then $u = v$ on $[0, T]$.

(iii) (Continuous dependency) the solution map $\Phi : L^q_{s_c}(\mathbb{R}^d) \to C([0,T)\,;\, L^q_{s_c}(\mathbb{R}^d)) \cap \mathcal{K}^s(T)$ is Lipschitz continuous.

(iv) (Blow-up criterion) If u is an $L^q_{s_c}(\mathbb{R}^d)$-mild solution constructed in the assertion (i) and $T_m < \infty$, then $\|u\|_{\mathcal{K}^s(T_m)} = \infty$.

(v) (SDGE and dissipation) There exists $\varepsilon_0 > 0$ depending only on d, γ, α, q and s such that if $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ satisfies $\|e^{t\Delta}u_0\|_{\mathcal{K}^s} < \varepsilon_0$, then $T_m = \infty$ and $\|u\|_{\mathcal{K}^s} \le 2\varepsilon_0$. Moreover, the solution u is dissipative. In particular, if $\|u_0\|_{L^p_{s_c}}$ is sufficiently small, then $\|e^{t\Delta}u_0\|_{\mathcal{K}^s} < \varepsilon_0$.

Sketch of proof

Key estimate (Kobayashi-Kubo(2013), Tsutsui (2014), others)

Let $d \geq 1$, $1 \leq p \leq q \leq \infty$, $-\frac{d}{q} < s' \leq s < d\left(1 - \frac{1}{p}\right) \cdots (C)$. In addition, $s \leq 0$ when $p = 1$ and $0 \leq s'$ when $q = \infty$. Then

$$\|e^{t\Delta} f\|_{L^q_{s'}} \leq C t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-\frac{s-s'}{2}} \|f\|_{L^p_s}.$$

Moreover, condition (C) is optimal.

Sketch of proof of Optimality Optimality of $s < d\left(1 - \frac{1}{p}\right)$: Let $d\left(1 - \frac{1}{p}\right) < s$.

$$f(x) := \begin{cases} |x|^{-d}, & |x| \leq 1 \\ 0, & \text{else,} \end{cases} \in L^p_s(\mathbb{R}^d) \quad \because\ p(d-s) < d$$

but $f \notin L^1_{loc}(\mathbb{R}^d)$; $e^{t\Delta} f$ does not make sense. (Same argument for $s = d\left(1 - \frac{1}{p}\right)$)

Optimality of $-\frac{d}{q} < s'$: Suppose that the inequality holds when $s' \le -\frac{d}{q}$. We have $g \in L^q_{s'}(\mathbb{R}^d) \Longrightarrow \liminf_{|x|\to 0} |x|^{\frac{d}{q}+s'}|g(x)| = 0$. In particular, $\liminf_{|x|\to 0} |g(x)| = 0$.

$$f(x) := \begin{cases} C, & |x| \le 1 \\ 0, & \text{else}, \end{cases} \in L^p_s(\mathbb{R}^d) \quad \because\ 0 < \frac{d}{p} + s < d$$

However, clearly $\liminf_{|x|\to 0} |e^{t\Delta} f(x)| \ne 0$. $\therefore\ e^{t\Delta} f \notin L^q_{s'}(\mathbb{R}^d)$.

Sketch of proof of LWP (i) (Existence)

- Banach's fixed point theorem allows us to find a solution in $\mathcal{K}^s(T)$.
- By using the $\mathcal{K}^s(T)$-regularity, we then show $u \in C([0,T)\,;\, L^q_{s_c}(\mathbb{R}^d))$.

Sketch of proof

Well-definedness of $\mathcal{K}^s(T)$:

- We only show the estimate for the Duhamel term. Let $\sigma := \alpha s - \gamma$. By $L^{\frac{q}{\alpha}}_\sigma$-$L^q_s$ estimate,

$$\|N(u)(t)\|_{L^q_s} \le \int_0^t \|e^{(t-\tau)\Delta}\{|\cdot|^\gamma F(u(\tau))\}\|_{L^q_s} d\tau$$
$$\le C\int_0^t (t-\tau)^{-\frac{d(\alpha-1)}{2q}-\frac{1}{2}\{(\alpha-1)s-\gamma\}} \||\cdot|^\gamma F(u(\tau))\|_{L^{\frac{q}{\alpha}}_\sigma} d\tau$$

- $\||\cdot|^\gamma F(u)\|_{L^{\frac{q}{\alpha}}_\sigma} = \||\cdot|^{\sigma+\gamma} F(u(\tau))\|_{L^{\frac{q}{\alpha}}} = \||\cdot|^{\alpha s}|u|^\alpha\|_{L^{\frac{q}{\alpha}}} = \||\cdot|^s|u|\|^\alpha_{L^q} = \|u\|^\alpha_{L^q_s}$

$$\therefore \quad \|N(u)(t)\|_{L^q_s} \le Ct^{-\frac{s_c-s}{2}}\|u\|^\alpha_{\mathcal{K}^s(T)}.$$

□

Remark In known results the crux of the matter has been the handling of $|x|^\gamma$.

- If $\gamma < 0$, then one may see the it as $|x|^\gamma \in L^{\frac{d}{-\gamma},\infty}(\mathbb{R}^d)$ (Ben Slimene et al. (2017)) $|x|^\gamma \in \dot{B}^{\frac{d}{p}+\gamma}_{p,\infty}(\mathbb{R}^d)$ (Chikami (2019))
- In this work, we treat $|x|^\gamma$ as an increase/decrease of the weights in $L^q_s(\mathbb{R}^d)$-norms, allowing us to treat the Hénon case $\gamma > 0$.

Forward self-similar solution

Forward self-similar solutions: A forward self-similar solution is a solution such that $u_\lambda = u$ for all $\lambda > 0$, where $u_\lambda = \lambda^{\frac{2+\gamma}{\alpha-1}} u(\lambda^2 t, \lambda x)$.

Existence of forward self-similar solutions

Let $d \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ satisfy $\alpha > 1 + \frac{2+\gamma}{d}$. Let $\varphi(x) := \omega(x)|x|^{-\frac{2+\gamma}{\alpha-1}}$, where $\omega \in L^\infty(\mathbb{R}^d)$ is homogeneous of degree 0 and $\|\omega\|_{L^\infty}$ is sufficiently small so that $\|e^{t\Delta}\varphi\|_{\mathcal{K}^s} < \varepsilon_0$. Then there exists a self-similar solution $u_{\mathcal{S}}$ of (HH) with the initial data φ such that $u_{\mathcal{S}}(t) \to \varphi$ in $\mathcal{S}'(\mathbb{R}^d)$ as $t \to 0$.

Known results

- Wang (1993) Radially symmetric self-similar solutions for $d \geq 3$, $\gamma > -2$ and $\alpha \geq 1 + \frac{2(2+\gamma)}{d-2}$.
- Hirose (2008) Radially symmetric self-similar solutions for $d \geq 3$, $1 + \frac{2+\gamma}{d} < \alpha < 1 + \frac{2(2+\gamma)}{d-2}$, $\gamma \leq 0$ for $d \geq 4$ and $\gamma \leq \sqrt{3} - 1$ for $d = 3$.
- Ben Slimene et al. (2017) Non-radially symmetric self-similar solutions for $\alpha > 1 + \frac{2+\gamma}{d}$, but only for the Hardy case $\gamma < 0$.

Weak solution $T > 0$. We call a function $u : [0,T) \times \mathbb{R}^d \to \mathbb{R}$ a weak solution to (HH) if u belongs to $L^\alpha(0,T; L^\alpha_{\frac{\gamma}{\alpha},loc}(\mathbb{R}^d))$ and if it satisfies the equation (HH) in the distributional sense, i.e.,

$$\int_{\mathbb{R}^d} u(T',x)\eta(T',x)\,dx - \int_{\mathbb{R}^d} u_0(x)\eta(0,x)\,dx = \int_{[0,T']\times\mathbb{R}^d} u(t,x)(\Delta\eta + \eta_t)(t,x) + |x|^\gamma u(t,x)^\alpha\,\eta(t,x)\,dx\,dt \tag{1}$$

for all $T' \in [0,T]$ and for all $\eta \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $\operatorname{supp}\eta(t,\cdot)$ is compact.

Nonexistence for supercritical data : $s > s_c$

Let $d \in \mathbb{N}$ and $\gamma \in \mathbb{R}$. Assume that $q \in [1,\infty]$, $\alpha \in \mathbb{R}$ and $s \in \mathbb{R}$ satisfy $\alpha > \max(1, \alpha_F(d,\gamma))$ and $s > s_c$. Then there exists an initial data $u_0 \in L^q_s(\mathbb{R}^d)$ such that the problem (HH) with $u(0) = u_0$ has no local positive weak solution.

Sketch of proof

- Test function method. (cf. Ikeda-Inui (2015)).

Sketch of test function method

$$\eta(t) := \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ 0, & t \ge 1, \end{cases} \quad \text{and} \quad \phi(x) := \begin{cases} 1, & |x| \le \frac{1}{2}, \\ 0, & |x| \ge 1. \end{cases}$$

Let $\psi_T(t,x) := \eta\left(\frac{t}{T}\right)\phi\left(\frac{x}{\sqrt{T}}\right)$ Multiplying (HH) by ψ_T^l ($l \in \mathbb{Z}_+$) and integrating over $[0,T) \times \{|x| < \sqrt{T}\}$ we obtain

$$\int_{[0,T)\times\{|x|<\sqrt{T}\}} \partial_t u \psi_T^l \, dtdx - \int_{[0,T)\times\{|x|<\sqrt{T}\}} \Delta u \psi_T^l \, dtdx = \int_{[0,T)\times\{|x|<\sqrt{T}\}} |x|^\gamma |u|^{\alpha-1} u \psi_T^l \, dtdx =: I(T).$$

By Integration by parts, $|\partial_t \psi_T^l(t,x)| \le \frac{C}{T}\psi_T^{l-1}$ and $|\partial_{x_j}^2 \psi_T^l(t,x)| \le \frac{C}{T}\psi_T^{l-1}$,

$$I(T) + \int_{|x|<\sqrt{T}} u_0(x)\phi^l\left(\frac{x}{\sqrt{T}}\right) dx = -\int_{[0,T)\times\{|x|<\sqrt{T}\}} u(\partial_t\psi_T^l + \Delta\psi_T^l)\, dtdx \le \frac{C}{T}\int_{[0,T)\times\{|x|<\sqrt{T}\}} |u| \psi_T^{\frac{l}{\alpha}} \, dtdx,$$

Then there exists $C > 0$ independent of u and T such that

$$\int_{|x|<\sqrt{T}} u_0(x)\phi^l\left(\frac{x}{\sqrt{T}}\right) dx \leq \frac{C}{T}\int_{[0,T)\times\{|x|<\sqrt{T}\}} |u|\psi_T^{\frac{l}{\alpha}}\, dtdx - I(T)$$

$$\leq \cdots \text{(Hölder and Young's inequalities)} \cdots$$

$$\leq CT^{-\alpha'+1}\int_{|x|<\sqrt{T}} |x|^{-\frac{\gamma}{\alpha-1}}\, dx$$

$$\leq CT^{-\frac{2+\gamma}{2(\alpha-1)}+\frac{d}{2}}.$$

Let

$$u_0(x) = \begin{cases} |x|^{-\beta} & |x| \leq 1, \\ 0 & \text{otherwise} \end{cases} \in L^q_s(\mathbb{R}^d) \quad \left(\frac{2+\gamma}{\alpha-1} < \beta < s + \frac{d}{q}\right)$$

$$\Longrightarrow \int_{|x|<\sqrt{T}} u_0(x)\phi^l\left(\frac{x}{\sqrt{T}}\right) dx = CT^{-\frac{\beta-d}{2}}.$$

$$\Longrightarrow 0 < C \leq T^{\frac{\beta}{2}-\frac{2+\gamma}{2(\alpha-1)}} \to 0 \quad \text{as } T \to 0.$$

□

Summary

1. (Hardy-Hénon parabolic equation)

$$\begin{cases} \partial_t u - \Delta u = |x|^\gamma |u|^{\alpha-1} u, \quad t > 0,\ x \in \mathbb{R}^d, \\ u|_{t=0} = u_0 \in L^q_{s_c}(\mathbb{R}^d). \end{cases}$$

2. LWP in weighted spaces $L^q_s(\mathbb{R}^d)$
3. (Main results) $\gamma > -\min(2,d)$, $\alpha > 1 + \frac{2+\gamma}{d}$
   - (i) If $s \le s_c$ then (HH) is LWP $L^q_s(\mathbb{R}^d)$.
   - (ii) If $s > s_c$ then there exists $u_0 \in L^q_s(\mathbb{R}^d)$ such that (HH) with $u(0) = u_0$ does not have a weak solution.
4. (Further possible works in relation to Harmonic Analysis)
   - Compactness/noncompactness properties of the space $L^q_{s,rad}(\mathbb{R}^d)$.
   - Global dynamics of the equation in energy spaces.
   - Analysis in hyperbolic spaces.

Thank you for your attention!