Global theory of subelliptic pseudo-differential operators and Fourier integral operators on compact Lie groups.

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- Pseudo-differential operators

Joint work with Prof. Dr. Michael Ruzhansky

Outline

1. Introduction
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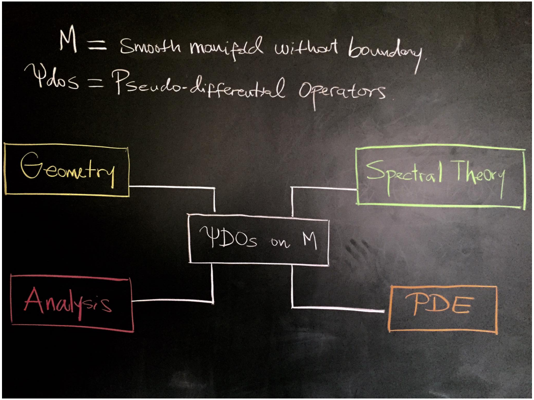
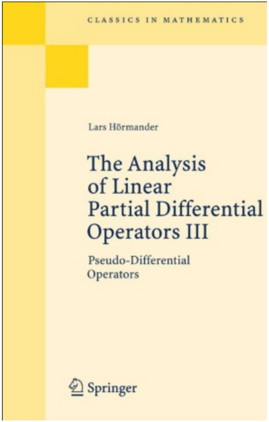
3. Main results

Theory of pseudo-differential operators using local coordinate systems
Kohn-Nirenberg 1965, Hörmander 1967, etc.



Standard theory of pseudo-differential operators



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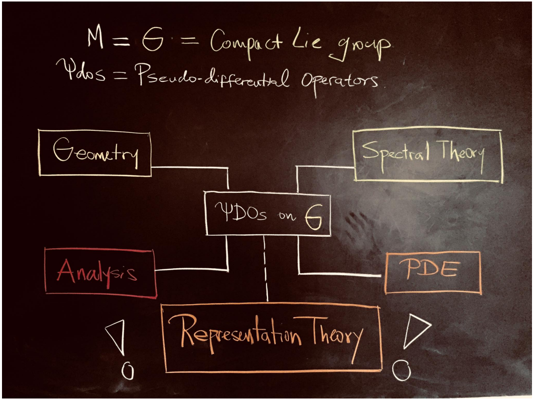
M. Ruzhansky, V. Turunen, J. Wirth.



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Pseudo-differential operator theory for sub-Riemannian structures on compact Lie groups

SUBELLIPTIC PSEUDO-DIFFERENTIAL OPERATORS AND FOURIER INTEGRAL OPERATORS ON COMPACT LIE GROUPS

DUVÁN CARDONA AND MICHAEL RUZHANSKY

ABSTRACT. In this work we extend the theory of global pseudo-differential operators on compact Lie groups to a subelliptic context. More precisely, given a compact Lie group G, and the sub-Laplacian $\mathcal{L}$ associated to a system of vector fields $X = \{X_1, \cdots, X_k\}$ satisfying the Hörmander condition, we introduce a (subelliptic) pseudo-differential calculus associated to $\mathcal{L}$, based on the matrix-valued quantisation process developed in [111]. This theory will be developed as follows. First, we will investigate the singular kernels of this calculus, estimates of L^p-L^p, H^1-L^1, L^∞-BMO type and also the weak (1,1) boundedness of these subelliptic Hörmander classes. Between the obtained estimates we prove subelliptic versions of the celebrated sharp Fefferman L^p-theorem and the Calderón-Vaillancourt theorem. The obtained estimates will be used to establish the boundedness of subelliptic operators on subelliptic Sobolev and Besov spaces. We will investigate the ellipticity, the construction of parametrices, the heat traces and the regularisation of traces for the developed subelliptic calculus. A subelliptic global functional calculus will be established as well as a subelliptic version of Hulanicki theorem. This subelliptic functional calculus will be used to prove a subelliptic version of the Gårding inequality, which we also use to study the global solvability for a class of subelliptic pseudo-differential problems. Finally, by using both, the matrix-valued symbols and also the notion of matrix-valued phases we study the L^2-boundedness of global Fourier integral operators. The approach established in characterising our subelliptic Hörmander classes (by proving that the definition of these classes is independent of certain parameters) will be also applied in order to characterise the global Hörmander classes on arbitrary graded Lie groups developed in [66].

CONTENTS

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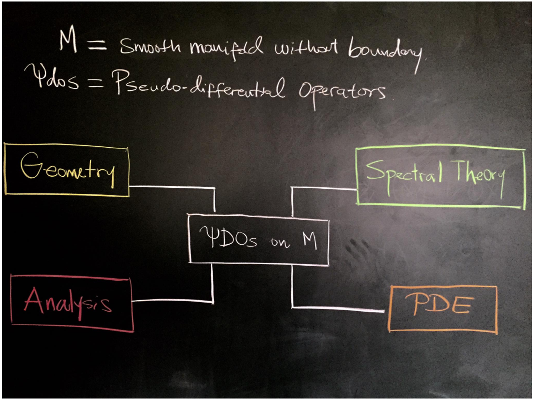
1. Introduction and historical remarks

This work is devoted to the development of the pseudo-differential calculus for subelliptic pseudo-differential operators on arbitrary compact Lie groups and its applications. For instance, the theory developed here could remains valid in several compact non-commutative structures with the presence of symmetries (see [111, Part IV]).

In modern mathematics, the theory of pseudo-differential operators is a powerful branch in the analysis of linear partial differential operators due to its interactions with several areas of mathematics. For instance, from the point of view of the theory of partial differential equations, pseudo-differential operators are used e.g. to study the global/local solvability of several partial differential problems, to understand the mapping properties of certain singular integral operators, to understand the propagation of singularities in distribution theory, and in the construction of fundamental solutions and parametrices. Also, in the interplay between differential geometry and algebraic topology, pseudo-differential operators are used to compute some geometric invariants arising in the index theory. That is the case of analytical expressions for the Euler characteristic, the Hirzebruch signature and, in a more general context, the Atiyah-Singer index theorem (see e.g. Atiyah and Singer [6, 7, 8, 9, 10, 11], Kohn and Nirenberg [85], the fundamental book by Hörmander [81] and references therein). On the other hand, in the microlocal analysis, the theory of Fourier integral operators becomes a prominent generalisation of pseudo-differential operators, to study the spectral function for elliptic operators on vector bundles and in solving hyperbolic differential equations (see Duistermaat and Hörmander [57] and Hörmander [80]).

In this work we develop a subelliptic pseudo-differential calculus on compact Lie groups and some of its applications, by contributing with the notions and results of harmonic analysis on compact Lie groups, building up on the monograph [111] by V. Turunen and the second author, which was devoted to the development of the general theory of global pseudo-differential operators (with matrix-valued symbols) on spaces with symmetries. Starting our work, we investigate the action of the subelliptic calculus on L^p, subelliptic Sobolev, and subelliptic Besov spaces and in the final part of the paper, we study the L^2-boundedness of global Fourier integral operators. We will follow the notion of global symbol on every compact Lie group G introduced in [111] which is a non-commutative extension of the classical Konh-Nirenberg quantisation [85], instead of the notion of a symbol

Let me introduce some preliminaries of the standard theory...



Hörmander classes of symbols on $M = U \subset \mathbb{R}^n$.

- If $U \subset \mathbb{R}^n$ is open, $U \neq \mathbb{R}^n$, the symbol $a : U \times \mathbb{R}^n \to \mathbb{C}$, belongs to the Hörmander class $S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, $0 \leqslant \rho, \delta \leqslant 1,$ if for every compact subset $K \subset U$, the symbol inequalities,

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant C_{\alpha,\beta,K}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

  hold true uniformly in $x \in K$ and $\xi \in \mathbb{R}^n$.
- For $U = \mathbb{R}^n$, the definition is similar requesting estimates uniformly in $x \in \mathbb{R}^n$.

Analysis. Pseudo-differential operators on $M = U \subset \mathbb{R}^n$.

- A continuous linear operator $A : C_0^\infty(U) \to C^\infty(U)$ is a pseudo-differential operator of order m, of (ρ, δ)-type, if there exists a function $a \in S^m_{\rho,\delta}(U \times \mathbb{R}^n)$, satisfying

$$Af(x) = \int\limits_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} a(x,\xi)(\mathscr{F}_{\mathbb{R}^n} f)(\xi) d\xi,$$

for all $f \in C_0^\infty(U)$, where

$$(\mathscr{F}_{\mathbb{R}^n} f)(\xi) := \int\limits_U e^{-i2\pi x\cdot\xi} f(x) dx,$$

is the Euclidean Fourier transform of f at $\xi \in \mathbb{R}^n$.

Analysis. Pseudo-differential operators on a closed manifold M.

- The class $S^m_{\rho,\delta}(U\times\mathbb{R}^n)$ on the phase space $U\times\mathbb{R}^n$, is invariant under coordinate changes only if $\rho\geqslant 1-\delta$, while a symbolic calculus (closed for products, adjoints, parametrices, etc.) is only possible for $\delta<\rho$ and $\rho\geqslant 1-\delta$.
- $A:C^\infty_0(M)\rightarrow C^\infty(M)$ is a pseudo-differential operator of order m, of (ρ,δ)-type, $\rho\geqslant 1-\delta$, if for every local coordinate patch $\omega:M_\omega\subset M\rightarrow U\subset\mathbb{R}^n$, and for every $\phi,\psi\in C^\infty_0(U)$, the operator

$$Tu:=\psi(\omega^{-1})^*A\omega^*(\phi u),\ u\in C^\infty(U),$$ [1]

is a pseudo-differential operator with symbol in $S^m_{\rho,\delta}(U\times\mathbb{R}^n)$.

[1] As usually, ω^* and $(\omega^{-1})^*$ are the pullbacks induced by the maps ω and ω^{-1}, respectively.

Analysis. Pseudo-differential operators on a closed manifold M.

- In this case we write that $A \in \Psi^m_{\rho,\delta}(M, \text{loc})$, $\delta < \rho,\ \rho \geqslant 1-\delta.$
- To $A \in \Psi^m_{\rho,\delta}(M; \text{loc})$ one associates a (principal) symbol $a \in S^m_{\rho,\delta}(T^*M)$,[2] which is uniquely determined, only as an element of the quotient algebra $S^m_{\rho,\delta}(T^*M)/S^{m-1}_{\rho,\delta}(T^*M)$.

(Q): When, is it possible to define a notion of a global symbol (without using local coordinate systems) allowing a global quantisation formula for the Hörmander class $\Psi^m_{\rho,\delta}(M; \text{loc})$?

[2] which is a section of the cotangent bundle T^*M.

Analysis. Boundedness properties of pseudo-differential operators

- (Calderón-Vaillancourt Theorem). $T_\sigma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is bounded if $m = 0$ and $1 \leq \delta \leq \rho \leq 1$, $\delta < 1$.[3]

- (Fefferman L^p-Theorem). Let $1 \leq \delta < \rho \leq 1$, and let $1 < p < \infty$. $T_\sigma : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded, for all $T_\sigma \in \Psi^{-m,\mathcal{L}}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)\}$, if $m \geq n(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$.[4]

[3] Calderón, A. P., Vaillancourt, R. A class of bounded pseudo-differential operators, Proc. Nat. Acad. Sci. USA 69, pp. 1185–1187, (1972).

[4] Fefferman, C., L^p-bounds for pseudo-differential operators, Israel J. Math. 14, pp. 413–417, (1973).

Spectral Theory. Traces and residues. Used to compute topological invariants

- Analytic functional calculus[5] and spectral functional calculus of elliptic operators.

$$F(A) = -\frac{1}{2\pi i}\oint_{\partial\Lambda} F(z)(A - zI)^{-1}dz, \quad \tilde{F}(T) = \int_0^\infty \tilde{F}(\lambda)dE_\lambda. \tag{0.1}$$

- $\mathrm{Tr}(Ae^{-t\Delta})$, $\mathrm{ind}(A) = \mathrm{Tr}(e^{-tA^*A}) - \mathrm{Tr}(e^{-tAA^*})$.[6]
- $\mathrm{res}_{z=0}\mathrm{Tr}(A\Delta^{-z/2})$.

[5] Seeley, R. T. The resolvent of an elliptic boundary problem, Amer. J. Math., (1969), 91, 889–920.

[6] Gilkey, P. Invariance theory, the equation and the Atiyah–Singer index theorem, Publish or Perish, Wilmington, 1984.

PDE

- Gårding inequality for elliptic operators[7]

$$(Pu,u)\geq c_{\mu,K}\|u\|^2_{H^{\frac{m}{2}}}-C_{\mu,K}\|u\|^2_{H^{\mu}},\quad \forall u\in C_0^\infty(K),\ \mu<m. \tag{0.2}$$

- Sharp Gårding inequality, $\mathrm{Re}(p(x,\xi))\geq 0,$[8]

$$\mathrm{Re}(Pu,u)\geq -C_K\|u\|^2_{\frac{m-1}{2}},\quad \forall u\in C_0^\infty(K). \tag{0.3}$$

- Well-posedness for the Cauchy problem,

$$(\mathrm{PVI}):\begin{cases}\frac{\partial v}{\partial t}=P(t,x,D)v+f, & v\in\mathscr{D}'((0,T)\times G),\\ v(0)=u_0.\end{cases} \tag{0.4}$$

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[7] Gårding, L. *Dirichlet's problem for linear elliptic partial differential equations,* Math. Scand. (1) (1953), 55–72.

[8] Hörmander, L. *Pseudo-differential operators and non-elliptic boundary problems,* Ann. of Math. 83(2) (1966), 129–209.

Conclusion:

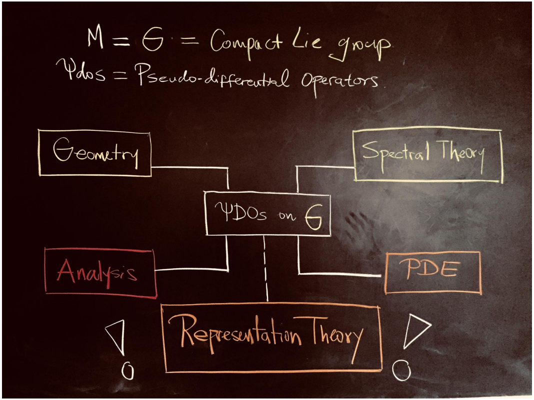
- To $A \in \Psi^m_{\rho,\delta}(M; \text{loc})$ one associates a (principal) symbol $a \in S^m_{\rho,\delta}(T^*M)$, which is uniquely determined, only as an element of the quotient algebra $S^m_{\rho,\delta}(T^*M)/S^{m-1}_{\rho,\delta}(T^*M)$.

(Q): When, is it possible to define a notion of a global symbol (without using local coordinate systems) allowing a global quantisation formula for the Hörmander class $\Psi^m_{\rho,\delta}(M; \text{loc})$?[9][10]

[9] Fulling, S. A. Pseudodifferential operators, covariant quantization, the inescapable Van Vleck-Morette determinant, and the $R/6$ controversy. The Sixth Moscow Quantum Gravity Seminar (1995). Internat. J. Modern Phys. D 5, no. 6, 597–608, 1996.

[10] Fulling, S. A. Pseudodifferential operators, covariant quantization, the inescapable Van Vleck-Morette determinant, and the R/6 controversy. Relativity, particle physics and cosmology (College Station, TX, 1998), 329–342, World Sci. Publ., River Edge, NJ, 1999.

Global (Riemannian and Sub-Riemannian) theory of pseudo-differential operators on compact Lie groups



2. Preliminaries.

Notations.

- $M = G$ is a compact Lie group.
- ξ denotes a unitary representation of G, i.e. ξ is a continuous mapping

$$\xi \in \mathsf{HOM}(G, U(H_\xi)),\ \ \xi(x)\xi(y) = \xi(xy),\, \xi(x)^* = \xi(x)^{-1},$$

for some (finite-dimensional) vector space $H = H_\xi$. We define by $d_\xi = dim(H_\xi)$ the dimension of ξ.
- Equivalent Representations: Two representations

$$\xi \in \mathsf{HOM}(G, U(H_\xi)),\, \eta \in \mathsf{HOM}(G, U(H_\eta))$$

are equivalent, if there exist a linear bijection $\phi : H_\xi \to H_\eta$, such that $\forall x \in G,\ \xi(x) = \phi^{-1}\eta(x)\phi$.
- $\widehat{G}$ consists of all equivalence classes of continuous irreducible unitary representations of G.

Notations.

- Fourier transform of $f \in C^\infty(G)$,

$$(\mathscr{F} f)(\xi) \equiv \widehat{f}(\xi) := \int_G f(x)\xi(x)^* dx \in \mathbb{C}^{d_\xi \times d_\xi}, \quad [\xi] \in \widehat{G}. \tag{0.5}$$

- Fourier inversion formula,

$$f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\widehat{f}(\xi)). \tag{0.6}$$

Continuous Linear operators on G and the Fourier Transform. We write $\xi(x)=[\xi_{ij}(x)]_{i,j=1}^{d_\xi}\in\mathbb{C}^{d_\xi\times d_\xi}$.

Theorem (Ruzhansky-Turunen)

Let $A:C^\infty(G)\to C^\infty(G)$ be a continuous linear operator. Then:

$$Af(x)=\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma(x,\xi)(\mathscr{F}f)(\xi)]\ \ f\in C^\infty(G),$$

where

$$\sigma(x,\xi):=\xi(x)^*A\xi(x):=\xi(x)^*[A\xi_{ij}(x)]_{i,j=1}^{d_\xi}.$$

Examples.

- The symbol of the Laplacian $\mathcal{L}_G = -\sum_{i=1}^{n} X_i^2$. Observe that

$$\mathcal{L}_G \xi(x) = \lambda_{[\xi]} \xi(x),$$

where $\mathcal{L}_G$ is the positive Laplacian on the group G. The symbol of the positive Laplacian is given by

$$\sigma_{\mathcal{L}_G}(x, \xi) = \lambda_{[\xi]} I_{H_\xi}.$$

Examples.

- The symbol of the sub-Laplacian $\mathcal{L}_X = -X_1^2 - \cdots - X_k^2$, where $X = \{X_1, \cdots, X_k\}$ satisfies the Hörmander condition at step κ.
- The symbol of the positive sub-Laplacian is given by

$$\sigma_{\mathcal{L}}(x,\xi) = \widehat{\mathcal{L}}(\xi) := \text{diag}[\nu_{ii}^2(\xi)].$$

$$(1+\lambda_{[\xi]})^{\frac{1}{2\kappa}} \lesssim \nu_{ii}(\xi) \lesssim (1+\lambda_{[\xi]})^{\frac{1}{2}}.$$ [11]

- Definition: (Matrix-valued subelliptic weight)

$$\mathcal{M}(\xi) := (1+\widehat{\mathcal{L}}(\xi))^{\frac{1}{2}}.$$

[11] Garetto, C., Ruzhansky, M. Wave equation for sum of squares on compact Lie groups, J. Differential Equations. 258, pp. 4324–4347, 2015.

Hörmander classes for the sub-Riemannian structure associated with the Hörmander system $X=\{X_1,\cdots,X_k\}$.

Let us consider the sub-Laplacian $\mathcal{L}=-\sum_{i=1}^{k}X_i^2$.

- **Definition. Subelliptic Hörmander classes:** Let $0\leq\delta,\rho\leq 1$. Then $a\in S^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$, if,

$$\sup_{(x,[\xi])\in G\times\widehat{G}}\|\widehat{\mathcal{M}}(\xi)^{(\rho|\alpha|-\delta|\beta|-m)}\partial_x^\beta\Delta_\xi^\alpha a(x,\xi)\|_{\text{op}}<\infty.$$

Difference operators:

$$\Delta_\xi^\alpha a(x,\xi):=\widehat{q_\alpha k_x},$$

where $q_\alpha(x)\sim|x|^{|\alpha|}$, and $a(x,\xi)=\widehat{k}_x(\xi)$.

Subelliptic pseudo-differential calculus on G. Stability of the theory under compositions and adjoints.

Define

$$\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G}):=\{T_\sigma:\sigma\in S^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})\},$$

for $0\leq\delta\leq\rho\leq 1$. Then:

- If $T_\sigma\in\Psi^{m_1,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ and $T_\tau\in\Psi^{m_2,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$, then, $T_\sigma\circ T_\tau\in T_\sigma\in\Psi^{m_1+m_2,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$,
- If $T_\sigma\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$ then $T_\sigma^*\in\Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$.

3. Main results. Applications in Analysis, Spectral theory, PDE, and Geometry.

Some results related to the analysis of subelliptic operators: mapping properties.

- (Calderón-Vaillancourt Theorem). $T_\sigma : L^2(G) \to L^2(G)$ is bounded if $m = 0$ and $1 \leq \delta \leq \rho \leq 1$, $\delta < 1/\kappa$.
- (Fefferman L^p-Theorem). Let $1 \leq \delta < \rho \leq 1$, and let $1 < p < \infty$. $T_\sigma : L^p(G) \to L^p(G)$ is bounded, for all $T_\sigma \in \Psi^{-m,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})\}$, if $m \geq Q(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$.

The subelliptic functional calculus on G.

- We prove a subelliptic version of the Hulanicki Theorem[12])
- The subelliptic calculus is stable under the spectral functional calculus of the sub-Laplacian:
  - Let $f \in S^{\frac{m}{2}}(\mathbb{R}_0^+)$, $m \in \mathbb{R}$, $(|\partial_\lambda^k f(\lambda)| \lesssim (1+\lambda)^{\frac{m}{2}-k})$. Then, for all $t > 0$,

$$f(t\mathcal{L}) = \int_0^\infty f(t\lambda)dE_\lambda \in S_{1,0}^{m,\mathcal{L}}(G \times \widehat{G}).$$

[12] Jerison, D., Sánchez-Calle, A. Subelliptic second order differential operators, in Complex analysis III, Springer, 1987, 46–77.

The subelliptic functional calculus on G.

- The subelliptic calculus is stable under the action of the complex functional calculus.

$$F(A) := -\frac{1}{2\pi i}\oint_{\partial\Lambda_\varepsilon} F(z)(A - zI)^{-1}dz,\ A = T_a{}^{13}. \qquad (0.7)$$

  - Let $m > 0$, and let $0 \leqslant \delta < \rho \leqslant 1$. Let $a \in S^{m,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})$ be a parameter $\mathcal{L}$-elliptic symbol with respect to Λ. Let us assume that F satisfies the estimate $|F(\lambda)| \leqslant C|\lambda|^s$ uniformly in λ,

[13] Ruzhansky, M., Wirth, J. Global functional calculus for operators on compact Lie groups, J. Funct. Anal., 267, 144–172, (2014).

for some $s \in \mathbb{R}$. Then $\sigma_{F(A)} \in S^{ms,\mathcal{L}}_{\rho,\delta}(G \times \widehat{G})$, for F satisfying some suitable conditions.

(CI). $\Lambda_\varepsilon := \Lambda \cup \{z : |z| \leqslant \varepsilon\}$, $\varepsilon > 0$, and $\Gamma = \partial\Lambda_\varepsilon \subset \text{Resolv}(A)$ is a positively oriented curve in the complex plane $\mathbb{C}$.

(CII). F is an holomorphic function in $\mathbb{C} \setminus \Lambda_\varepsilon$, and continuous on its closure.

(CIII). We will assume decay of F along $\partial\Lambda_\varepsilon$ in order that the operator (0.7) will be densely defined on $C^\infty(G)$ in the strong sense of the topology on $L^2(G)$.

Some results related to PDE

- (subelliptic Garding Inequality).

$$\mathrm{Re}(a(x,D)u,u) \geqslant C_1\|u\|_{L^{2,\mathcal{L}}_{\frac{m}{2}}(G)} - C_2\|u\|^2_{L^2(G)}.$$

- Well-posedness for the Cauchy problem

$$\text{(PVI)}: \begin{cases} \frac{\partial v}{\partial t} = P(t,x,D)v + f, \\ v(0) = u_0, v \in \mathscr{D}'((0,T)\times G) \end{cases} \tag{0.8}$$

- Sharp Gårding inequality, (Ruzhansky+C+Federico, 2021).

Some results related to Spectral theory

- Asymptotic expansions in spectral geometry

$$\mathbf{Tr}(A\psi(tE)) = t^{-\frac{Q+m}{q}}\left(\sum_{k=0}^{\infty} a_k t^k\right) + \frac{c_Q}{q}\int_0^{\infty}\psi(s)\times\frac{ds}{s}.$$

- $A \in \Psi^{m,\mathcal{L}}_{\rho,\delta}(G\times\widehat{G})$, $0\leqslant\delta,\rho\leqslant 1$, that

$$\mathbf{Tr}(Ae^{-t(1+\mathcal{L})^{\frac{q}{2}}}) \sim t^{-\frac{m+Q}{q}}\sum_{k=0}^{\infty} a_k t^{\frac{k}{q}} - \frac{b_0}{q}\log(t), \quad t\rightarrow 0^+, \tag{0.9}$$

for $m\geqslant -Q$. If $m=-Q$, then $a_k=0$ for every k, and for $m>-Q$, $b_0=0$.

Some results related to Noncommutative Geometry: Dixmier traces of subelliptic operators[14]

$$\mathbf{Tr}_{\omega}(A) = \int\limits_{G} \left(\|\mathrm{Re}(\sigma_{-Q}(x,[\xi]))^{+}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} - \|\mathrm{Re}(\sigma_{-Q}(x,[\xi]))^{-}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} \right) dx$$
$$+ i \int\limits_{G} \left(\|\mathrm{Im}(\sigma_{-Q}(x,[\xi]))^{+}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} - \|\mathrm{Im}(\sigma_{-Q}(x,[\xi]))^{-}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} \right) dx.$$

Notation: We have use the decomposition of bounded operators T, $\mathrm{Re}(T) := \frac{T+T^*}{2}$, $\mathrm{Im}(T) := \frac{T-T^*}{2i}$, and the decomposition of $\mathrm{Re}(T)$ and $\mathrm{Im}(T)$ into their positive and negative parts, $S^+ := \frac{S+|S|}{2}$, $S^- := \frac{|S|-S}{2}$, where $S = \mathrm{Re}(T), \mathrm{Im}(T)$.

[14] Cardona, D., Delgado, J., Ruzhansky, M. Dixmier traces, Wodzicki residues, and determinants on compact Lie groups: the paradigm of the global quantisation. submitted. arXiv:2105.14949

Dixmier ideal on Hilbert spaces

If $\mathcal{H}$ is a Hilbert space (we are interested in $\mathcal{H} = L^2(M)$ where M is a closed manifold of dimension n), the class $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ consists of those compact linear operators A on $\mathcal{H}$ satisfying

$$\sum_{1\leq n\leq N} s_n(A) = O(\log(N)), \quad N\to\infty, \tag{0.10}$$

where $\{s_n(A)\}$ denotes the sequence of singular values of A, i.e. the square roots of the eigenvalues of the positive-definite self-adjoint operator A^*A. So, $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ is endowed with the norm

$$\|A\|_{\mathcal{L}^{(1,\infty)}(\mathcal{H})} = \sup_{N\geq 2} \frac{1}{\log(N)} \sum_{1\leq n\leq N} s_n(A). \tag{0.11}$$

Some results related to Non-commutative Geometry: Wodzicki residues of classical operators[15]

$$\mathrm{res}(A) = \int_G \left(\|\mathrm{Re}(\sigma_{-n}(x,[\xi]))^{+}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} - \|\mathrm{Re}(\sigma_{-n}(x,[\xi]))^{-}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} \right) dx$$
$$+ i \int_G \left(\|\mathrm{Im}(\sigma_{-n}(x,[\xi]))^{+}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} - \|\mathrm{Im}(\sigma_{-n}(x,[\xi]))^{-}\|_{\mathcal{L}^{(1,\infty)}(\widehat{G})} \right) dx.$$

New description of residues in terms of representation theory:

$$\boxed{\mathrm{res}\,(A) \text{ is information encoded by the representation theory of } G} \tag{0.12}$$

[15] Cardona, D., Delgado, J., Ruzhansky, M. Dixmier traces, Wodzicki residues, and determinants on compact Lie groups: the paradigm of the global quantisation. submitted. arXiv:2105.14949

A classical pseudo-differential operator A on a closed manifold M of dimension n, has a symbol

$$\sigma^A(x,\xi) \sim \sum_{j=0}^{\infty} \sigma^A_{m-j}(x,\xi),\ \ (x,\xi) \in T^*M,$$

defined by localisations, and the Wodzicki residue, which measures the locality of the operator, is given by

$$\operatorname{res}(A) = \frac{1}{n(2\pi)^n} \int\limits_M \int\limits_{|\xi|=1} \sigma^A_{-n}(x,\xi)\, d\xi\, dx. \tag{0.13}$$

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