Spectrally invariant algebras of pseudodifferential operators with ultradifferentiable orbits

13th ISAAC conference

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2 August 2021

Introduction: The situation on $\mathbb{R}^n$

- The *Kohn-Nirenberg quantization* $\mathrm{Op}_{\mathbb{R}^n} : a \mapsto \mathrm{Op}_{\mathbb{R}^n}(a)$ is defined by

$$\mathrm{Op}_{\mathbb{R}^n}(a)\varphi(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i}\langle \xi, x\rangle} a(x,\xi)\widehat{\varphi}(\xi)\, \mathrm{d}\xi\,, \qquad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^n)\,,\ a \in \mathscr{S}(\mathbb{R}^{2n})\,.$$

- The Schrödinger representation of the Heisenberg group $\mathbb{H}$ is defined by

$$\rho(t,x,x') = \mathrm{e}^{2\pi \mathrm{i} t} T_{x/2} M_{x'} T_{x/2} \in \mathcal{L}(L^2(\mathbb{R}^n))\,,$$

  in which

$$T_x\varphi(y) = \varphi(y+x) \qquad \text{and} \qquad M_{x'}\varphi(y) = \mathrm{e}^{2\pi \mathrm{i}\langle y, x'\rangle}\varphi(y)$$

  for $(t,x,x') \in \mathbb{H} \simeq \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varphi \in L^2(\mathbb{R}^n)$.
- For the symbol class $\mathcal{S}^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n) = \{a \in C^\infty(\mathbb{R}^{2n}) \mid \forall_{\alpha\in\mathbb{N}_0^{2n}} : \partial^\alpha a \in L^\infty(\mathbb{R}^{2n})\}$:

Theorem (Beals, Cordes)

$a \in \mathcal{S}^0_{0,0}(\mathbb{R}^n\times\mathbb{R}^n)$ if and only if for $A := \mathrm{Op}_{\mathbb{R}^n}(a)$

$$z \mapsto \mathrm{Ad}_\rho(z)A := \rho(z)\,A\,\rho(z)^{-1} \text{ is in } \mathscr{E}(\mathbb{H}; \mathcal{L}(L^2(\mathbb{R}^n))).$$

Introduction: Situation on compact Lie groups G

- For a compact Lie group G the quantization $\mathrm{Op}_G \colon a \mapsto \mathrm{Op}_G(a)$ is defined by

$$\mathrm{Op}_G(a)\varphi(x) = \sum_{[\pi]\in\widehat{G}} \mathrm{Tr}[\pi(x)\, a(x,\pi)\, \widehat{\varphi}(\pi)] \cdot (\dim H_\xi)\,.$$

- $B^\infty(\widehat{G})$ is defined as the set of symbols $\sigma \in \prod_{\pi\in\mathrm{Rep}(G)} \mathcal{L}(H_\pi)$ with
  1. $\sigma(U\pi U^{-1}) = U\sigma(\pi)U^{-1}$ for each unitary U on H_π
  2. $\|\sigma\|_{B^\infty(\widehat{G})} := \sup_\pi \|\sigma(\pi)\|_{\mathcal{L}(H_\pi)} < \infty$
- Let $L_2(x)\varphi(y) = \varphi(x^{-1}y)$ for $\varphi \in L^2(G)$.

Theorem (Connolly, Fischer)

$a \in \mathscr{E}(G; B^\infty(\widehat{G}))$ iff $x \mapsto \mathrm{Ad}_{L_2}(x)A := L_2(x)AL_2(x)^{-1}$ is in $\mathscr{E}(G; \mathcal{L}(L^2(G)))$ for $A = \mathrm{Op}_G(a)$.

Here $\mathcal{S}^0_{0,0}(G \times \widehat{G}) = \mathscr{E}(G; B^\infty(\widehat{G}))$.

Theorem (Cabral, Melo)

$a \in \mathscr{A}(\mathbb{T}^n; B^\infty(\widehat{\mathbb{T}^n}))$ iff $x \mapsto \mathrm{Ad}_{L_2}(x)A = L_2(x)AL_2(x)^{-1}$ is in $\mathscr{A}(\mathbb{T}^n; \mathcal{L}(L^2(\mathbb{T}^n)))$ for $A = \mathrm{Op}_G(a)$ and the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Function spaces and vectors of generalized regularity

Definition

Let G be a Lie group. We call a continuously embedded subspace $\mathscr{F}(G) \subset \mathscr{C}(G)$ a $\mathscr{C}(G)$-function space.

For locally convex spaces E, F the product $E \,\varepsilon\, F$ is defined as

$$F \,\varepsilon\, E := \mathcal{L}_\varepsilon(E'_c; F)\,,$$

in which E'_c is the dual space of E equipped with the topology of uniform convergence on compact absolutely convex sets and $\mathcal{L}_\varepsilon(E'_c; F)$ is equipped with the topology of uniform convergence on equicontinuous subsets of E'_c.

Definition

If (π, E) is a representation of G on E, then define

$$\mathscr{F}(G;E) := \{f\colon G \to E \mid [e' \mapsto e' \circ f] \in \mathscr{F}(G) \,\varepsilon\, E\} \simeq \mathscr{F}(G) \,\varepsilon\, E$$
$$\text{and } \mathscr{F}(\pi) := \{e \in E \mid \pi(\cdot)e \in \mathscr{F}(G;E)\}$$

equipped with initial topology via the embedding $\mathscr{F}(\pi) \hookrightarrow \mathscr{F}(G;E)$.

Lemma

If $\mathscr{F}(G) = \varprojlim_\alpha \mathscr{F}_\alpha(G) = \bigcap_\alpha \mathscr{F}_\alpha(G)$, *then* $\mathscr{F}(\pi) = \varprojlim_\alpha \mathscr{F}_\alpha(\pi) = \bigcap_\alpha \mathscr{F}_\alpha(\pi)$.

Adjoint representations

1. (π, E) is said to be locally equicontinuous, if $\pi(K) \subset \mathcal{L}(E)$ is equicontinuous for any compact $K \subset G$.
2. (π, E) is locally equicontinuous and strongly continuous, iff $\mathscr{C}(\pi) = E$.

Definition

Ad_π is the representation of G on $\mathcal{L}_b(E)$ defined by $\mathrm{Ad}_\pi(x)T = \pi(x)T\pi(x)^{-1}$. Here $\mathcal{L}_b(E)$ carries the topology of uniform convergence on bounded sets.

3. If (π, E) is locally equicontinuous, then $(\mathrm{Ad}_\pi, \mathcal{L}_b(E))$ is locally equicontinuous.
4. For any leftinvariant differential operator P the operator $\pi(P)e := P_x\pi(x)e|_{x=1}$ on $\mathscr{E}(\pi)$ is well defined.
5. For a left invariant vector field X and $T \in \mathscr{E}(\mathrm{Ad}_\pi)$ the commutator $\mathrm{Ad}_\pi(X)T = \pi(X)T - T\pi(X) \in \mathcal{L}(E)$ is well defined.

Spaces of ultradifferentiable functions

Let $M \in \mathbb{R}_+^{\mathbb{N}_0}$ be a weight sequence with

$$M_0 = 1 \leq M_1\,, \quad \forall_{n\geq 1}\colon M_n^2 \leq M_{n-1}M_{n+1} \quad \text{and} \quad \sup_k \left(\frac{M_{k+1}}{M_k}\right)^{\frac{1}{k}} < \infty$$

Sporadically we will also use

$$\textbf{(nQA)} \quad \sum_n \frac{M_{n-1}}{nM_n} < \infty$$

Let $D = (D_1, \dots, D_n)$ be an analytic frame on G. Denote $D_0 := 1$,

$$S := \{a \in \{0, \dots, n\}^{\mathbb{N}} \mid |a| := \#\operatorname{supp} a < \infty\} \qquad \text{and} \qquad D^a := \cdots D_{a_3} D_{a_2} D_{a_1}$$

Definition

For regular compact $K \subset G$ (e.g. a compact submanifold of codimension 0 and smooth boundary) we define the Banach space

$$\mathscr{E}_D^M(K) := \left\{ f \in \mathscr{E}(K) \;\middle|\; \lim_{|a|\to\infty} \frac{\|D^a f\|_\infty}{M_{|a|}\,|a|!} = 0 \right\} \text{ with norm } \|f\|_{D,M} := \sup_{a\in S} \frac{\|D^a f\|_\infty}{M_{|a|}\,|a|!}\,.$$

And also the limit spaces

$$\mathscr{E}^{\{M\}}(K) := \varinjlim_{h\to 0} \mathscr{E}_{hD}^M(K)\,, \quad \mathscr{F}(G) := \varprojlim_{\text{reg. comp. } K} \mathscr{F}(K)\,, \quad \text{for } \mathscr{F} \in \{\mathscr{E}_D^M, \mathscr{E}^{\{M\}}\}$$

1. $\mathscr{E}^{\{M\}}(G)$ does not depend D (see also [Dasgupta, Ruzhansky, 2016]) and is nuclear and complete
2. If $\mathbb{1} := (1, 1, 1, \dots)$, then $\mathscr{E}^{\{\mathbb{1}\}}(G) = \mathscr{A}(G)$ is the space of analytic functions.

As a direct consequence of the work of [Komatsu, 1982] we have the following.

3. $\mathscr{E}^{\{M\}}(G) = \varprojlim_{\lambda \in \Lambda} \mathscr{E}_D^{\lambda M}(G)$ for $\Lambda := \{(c_0 \cdots c_n)_{n \in \mathbb{N}_0} \mid c_n \nearrow \infty\}$ as linear spaces.
4. If **(nQA)** holds, then the identity also holds for the topologies.

If E is complete, then the following holds.

5. If (π, E) is locally equicontinuous and $\mathcal{P}(E)$ the set of continuous seminorms on E and $D \subset \mathfrak{g}$ a basis of left invariant vector fields, then
$$\mathscr{E}_D^M(\pi) = \left\{ e \in \mathscr{E}(\pi) \;\middle|\; \forall_{p \in \mathcal{P}(E)} \lim_{\substack{|a| \to \infty \\ a \in S}} \frac{p(\pi(D^a)e)}{M_{|a|}\,|a|!} = 0 \right\}.$$
6. For $f \in \mathscr{C}(E)$ we have $f \in \mathscr{E}^{\{M\}}(G; E)$ iff $\forall_{e' \in E'}\; e' \circ f \in \mathscr{E}^{\{M\}}(G)$.
7. Also
$$\mathscr{E}^{\{M\}}(\pi) = \varprojlim_{\lambda \in \Lambda} \mathscr{E}_D^{\lambda M}(\pi)$$
as linear spaces. If M fulfils **(nQA)**, then the above holds topologically.

We have the following two easy lemmata.

Lemma

Suppose π is unitary on a Hilbert space E and the pointwise complex conjugation is continuous from $\mathscr{F}(G)$ to itself. Then $\mathscr{F}(\mathrm{Ad}_\pi)$ is a $*$-subalgebra of $\mathcal{L}(E)$.

Lemma

Let $A^\times$ be the group of invertible elements for an algebra A. Suppose $\mathscr{F}(G;\mathcal{L}_b(E))$ is an algebra for the pointwise multiplication. Furthermore, suppose $f \in \mathscr{F}(G;\mathcal{L}_b(E))^\times$, iff $f \in \mathscr{F}(G;\mathcal{L}_b(E))$ and $\forall_x \colon f(x) \in \mathcal{L}_b(E)^\times$. Then

$$\mathscr{F}(\mathrm{Ad}_\pi)^\times = \mathscr{F}(\mathrm{Ad}_\pi) \cap \mathcal{L}_b(E)^\times$$

The following Lemma is a consequence of a theorem from [Klotz, 2014].

Lemma

Suppose A is any Banach algebra and $f \in \mathscr{E}^{\{M\}}(G;A)$ fulfils $f(x) \in A^\times$ for all $x \in G$. Then $[x \mapsto f(x)^{-1}]$ is in $\mathscr{E}^{\{M\}}(G;A)$.

Proposition

If π is any unitary representation on a Hilbert space E, then $\mathscr{E}^{\{M\}}(\mathrm{Ad}_\pi)$ is a spectrally invariant $*$-subalgebra of $\mathcal{L}_b(E)$.

Spectrally invariant subalgebras with ultradifferential orbits cont.

Let R resp. L be the right resp. left translation on $\mathscr{C}(G)$. For any R-invariant $\mathscr{C}(G)$ space $\mathscr{F}(G)$, denote by $\pi\restriction_{\mathscr{F}}$ the restriction of π to $\mathscr{F}(\pi)$.

1. If $\mathscr{G}(G)$ is another $\mathscr{C}(G)$-space with $\mathscr{G}(G)=\mathscr{G}(R\restriction_{\mathscr{F}})$, then $\mathscr{G}(\pi)=\mathscr{G}(\pi\restriction_{\mathscr{F}})$.
2. If **(nQA)** holds, then $\mathscr{E}_D^{\{M\}}(G)=\mathscr{E}^{\{M\}}(R\restriction_{\mathscr{E}})=\mathscr{E}^{\{M\}}(R\restriction_{\mathscr{C}^k})=\mathscr{E}^{\{M\}}(L)$. If **(nQA)** does not hold, then the identity still holds in the sense of linear spaces.

Proposition

Suppose π is a representation on a Banach space E, then $\mathscr{E}^{\{\mathbb{1}\}}(\mathrm{Ad}_\pi)$ and thus $\mathscr{E}^{\{M\}}(\mathrm{Ad}_\pi)$ is dense in $\mathscr{E}(\mathrm{Ad}_\pi)$.

Proof.

1. For any k, $\mathrm{Ad}_\pi\restriction_{\mathscr{C}^k}$ is a strongly continuous representation on the Banach space $\mathscr{C}^k(\mathrm{Ad}_\pi)$.
2. Thus the analytic vectors $\mathscr{E}^{\{\mathbb{1}\}}(\mathrm{Ad}_\pi\restriction_{\mathscr{C}^k})$ are dense in $\mathscr{C}^k(\mathrm{Ad}_\pi)$ by [Nelson, 1959].
3. Now we use $\mathscr{E}^{\{\mathbb{1}\}}(\mathrm{Ad}_\pi\restriction_{\mathscr{C}^k})=\mathscr{E}^{\{\mathbb{1}\}}(\mathrm{Ad}_\pi)$ as linear spaces.
4. And we finish by using $\mathscr{E}(\mathrm{Ad}_\pi)=\varprojlim_k \mathscr{C}^k(\mathrm{Ad}_\pi)$.

□

Symbols of algebras with ultradifferentiable orbits (compact case)

Proposition

Suppose G is compact and L_2 is the left regular representation of G on $L^2(G)$. The Kohn-Nirenberg quantization induces a linear homeomorphism

$$\mathrm{Op}_G\colon \mathscr{F}(G; B^\infty(\widehat{G})) \to \mathscr{F}(\mathrm{Ad}_{L_2})$$

for any $\mathscr{C}(G)$-function space $\mathscr{F}(G)$ with $\mathscr{F}(G) = \mathscr{F}(R\restriction_{\mathscr{E}}) = \mathscr{F}(L)$.
This holds especially for $\mathscr{F}(G) = \mathscr{E}^{\{M\}}(G)$, if M fulfils (nQA). Without (nQA) and with $\mathscr{F}(G) = \mathscr{E}^{\{M\}}(G)$ the above is still a bijection.

Proof.

1. Let $\widetilde{L}$ be the left-translation on $\mathscr{C}(G; B^\infty(\widehat{G}))$.
2. Then Op_G is an isomorphism from $\mathscr{E}(\widetilde{L})$ onto $\mathscr{E}(\mathrm{Ad}_{L_2})$ (e.g. [Fischer, 2015])
3. Op_G restricts to an isomorphism $\mathscr{F}(\widetilde{L}\restriction_{\mathscr{E}}) \to \mathscr{F}(\mathrm{Ad}_{L_2}\restriction_{\mathscr{E}})$, since $\mathrm{Op}_G \widetilde{L} = \mathrm{Ad}_{L_2} \mathrm{Op}_G$.
4. By $\mathscr{F}(G) = \mathscr{F}(R\restriction_{\mathscr{E}})$ we have $\mathscr{F}(\widetilde{L}) = \mathscr{F}(\widetilde{L}\restriction_{\mathscr{E}})$ and $\mathscr{F}(\mathrm{Ad}_{L_2}) = \mathscr{F}(\mathrm{Ad}_{L_2}\restriction_{\mathscr{E}})$.
5. $\mathscr{F}(G) = \mathscr{F}(L)$ leads to $\mathscr{F}(G; B^\infty(\widehat{G})) = \mathscr{F}(\widetilde{L})$.
6. Finally, even if **(nQA)** does not hold, the $\mathscr{C}(G)$-function space $\mathscr{F}(G) := \varprojlim_{\lambda\in\Lambda} \mathscr{E}_D^{\lambda M}(G)$ can be used.

□

We define the spaces

$$\mathscr{E}^M_{\partial,b}(\mathbb{R}^{2n}) := \left\{ f \in \mathscr{E}(\mathbb{R}^{2n}) \;\middle|\; \lim_{|\alpha|\to\infty} \frac{\|\partial^\alpha f\|_\infty}{M_{|\alpha|}\,|\alpha|!} = 0 \right\}$$

$$\text{with norm} \quad \|f\|_M = \sup_{\alpha \in N_0^{2n}} \frac{\|\partial^\alpha f\|_\infty}{M_{|\alpha|}\,|\alpha|!}$$

and

$$\mathscr{E}_b^{\{M\}}(\mathbb{R}^{2n}) = \varprojlim_{\lambda\in\Lambda} \mathscr{E}^{\lambda M}_{\partial,b}(\mathbb{R}^{2n})$$

If L_b is the left translation on $\mathscr{C}_b(\mathbb{R}^{2n}) := \{f \in \mathscr{C}(\mathbb{R}^{2n}) \mid \|f\|_\infty < \infty\}$, then $\mathscr{E}_b^{\{M\}}(\mathbb{R}^{2n}) = \mathscr{E}^{\{M\}}(L_b)$. Similar to the compact case, we get the following proposition.

Proposition

If M fulfils (nQA), *then*

$$\mathrm{Op}_{\mathbb{R}^n} : \mathscr{E}_b^{\{M\}}(\mathbb{R}^{2n}) \to \mathscr{E}^{\{M\}}(\mathrm{Ad}_\rho)$$

is a linear homeomorphism. If (nQA) *does not hold, then the above is still a bijection.*

The following theorem is a variation of a Lemma due to [Schwartz, 1958].

Theorem

Let $\mathscr{F}$, $\mathscr{G}$, $\mathscr{H}$, E, F and G be complete locally convex spaces and let

$$u\colon \mathscr{F}\times\mathscr{G}\to\mathscr{H} \qquad \text{and} \qquad b\colon E\times F\to G$$

be bilinear. Suppose $\mathscr{H}$ is nuclear u is continuous and b is hypocontinuous. Then there is a hypocontinuous bilinear map

$$\tfrac{b}{u}\colon (\mathscr{H}\,\varepsilon\,E)\times(\mathscr{K}\,\varepsilon\,F)\to\mathscr{L}\,\varepsilon\,G, \quad \text{with} \quad \tfrac{b}{u}(S\otimes e, T\otimes f) = u(S,T)\otimes b(e,f).$$

If G or $\mathscr{L}$ has the approximation property, then $\tfrac{b}{u}$ is unique.

Proposition

Suppose $\mathscr{F}(G)$ is a complete $\mathscr{C}(G)$-space with continuous multiplication

$$\mathscr{E}^{\{M\}}(G)\times\mathscr{F}(G)\to\mathscr{F}(G)$$

and suppose (E,π) a representation on a Fréchet space E with the approximation property. Then

$$\mathscr{E}^{\{M\}}(\mathrm{Ad}_\pi)\to\mathcal{L}_b(\mathscr{F}(\pi))\colon\ T\mapsto T\restriction_{\mathscr{F}(\pi)}$$

is well defined and continuous.

Continuity properties of operators with ultradifferentiable orbits cont.

Proposition

Suppose $\mathscr{F}(G)$ is a complete $\mathscr{C}(G)$-space with continuous multiplication

$$\mathscr{E}^{\{M\}}(G) \times \mathscr{F}(G) \to \mathscr{F}(G)$$

and suppose (E, π) a representation on a Fréchet space E with the approximation property. Then

$$\mathscr{E}^{\{M\}}(\mathrm{Ad}_\pi) \to \mathcal{L}_b(\mathscr{F}(\pi)) \colon T \mapsto T \restriction_{\mathscr{F}(\pi)}$$

is well defined and continuous.

Suppose G is compact. The following spaces are examples for $\pi = L_2$ that work with the proposition above. Furthermore, in the following cases $\mathscr{F}(L_2)$ is dense in $L^2(G)$.

1. $\mathscr{F}(G) = \mathscr{C}^k(G)$ with $H^k(G) = \mathscr{C}^k(L_2)$
2. $\mathscr{F}(G) = \mathscr{E}(G)$ with $\mathscr{E}(G) = \mathscr{E}(L_2)$

Suppose D is a basis of left invariant vector fields, then $\tilde{D} := L_2(D)$ is a basis of right invariant vector fields. Let $\tilde{M}$ be another weight sequence.

3. If $\sup_k (M_k/\tilde{M}_k)^{\frac{1}{k}} < \infty$, then we may also use $\mathscr{F}(G) = \mathscr{E}^{\{\tilde{M}\}}(G)$. With **(nQA)**, we have $\mathscr{E}^{\{\tilde{M}\}}(G) = \mathscr{E}^{\{\tilde{M}\}}(L_2)$
4. If $(M_k/\tilde{M}_k)^{\frac{1}{k}} \to 0$, then we may also use $\mathscr{F}(G) = \mathscr{E}^M_D(G)$ with the Sobolev space $H^M_{\tilde{D}}(G) = \mathscr{E}^M_D(L_2)$ of functions f with $\lim_{|a|\to\infty} \frac{\|\tilde{D}^a f\|_2}{M_{|a|}\,|a|!} = 0$

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Thank you for your attention!