Boas-Type Theorems For The q-Bessel Fourier Transform

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Introduction

In the recent mathematical literature, we find many articles which deal with the theory of q-Fourier analysis associated with the q-Hankel transform. This theory was elaborated first by Koornwinder and R.F. Swarttouw in 1992 [8] and then by Fitouhi and Al in 2006 [4]. They were interested in q-analogue of different integral transformations. In connection with q-difference Bessel operator and with the basic Bessel functions, they introduced several generalized q-Fourier transforms. So, it is natural to look for the q-analogue of some well-known classical theorems.

Introduction

One of the classical topics in harmonic analysis and approximation theory consists in finding necessary and sufficient conditions on the Fourier coefficients of a function to belong to a generalised Lipschitz class. These types of results are, nowadays, known as Boas-type results, since it was R.P. Boas who in 1967 proved the first characterisation of this type [2]. Since then, this theory has been widely studied by several authors.

Introduction

In 2008, Moricz, F. [11-14] has studied the continuity and smoothness properties of a function f with absolutely convergent Fourier series. He gave the best possible sufficient conditions in terms of the Fourier coefficients of f which ensure the belonging of f either to one of the Lipschitz classes $Lip(\alpha)$ and $lip(\alpha)$ for some $0 < \alpha \leq 1$, or to one of the Zygmund classes $Zyg(\alpha)$ and $zyg(\alpha)$ for some $0 < \alpha \leq 2$.

Introduction

Tikhonov, (2006-2007) [16-17], considered the cases of cosine and sine series separately. Recently (2004 - 2015), S.S. Volosivets has published two papers [18-19], in which he has generalized all the previous results, and also these results are generalized to quaternion Fourier transform in 2021 by El. M. Loualid, A. Elgargati and R. Daher [10].

Introduction

The aim of this talk is to give a q-analogue of the aforesaid results in q-Bessel Fourier setting.

Introduction

This talk is arranged as follows:

★ In Section 2, we will state some basic notions and results from harmonic analysis related to the q-Bessel Fourier transform$\mathcal{F}_{q,\nu}$ that will be needed throughout this talk.

★ In Section 3, We will show some auxiliary results required to the proofs of our main results.

★ In section 4, after defining the generalized Lipschtiz classes $H^m_{q,\alpha}$ and $h^m_{q,\alpha}$, we will state the main theorems of this talk.

Preliminaries

In this section, we summarize some harmonic analysis tools related to the q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$ that will be used hereafter.
Throught this talk, we assume that $0 < q < 1$, $\nu > -1$ and

$$\mathbb{R}_q^+ = \{q^n, n \in \mathbb{Z}\}.$$

Preliminaries

Definition

Let $a \in \mathbb{C}$, the q-shifted factorial are defined by:

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

Definition

The q-derivative of a function f is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad if \quad x \neq 0$$

Definition

The q-Jackson integrals from 0 to a and from 0 to ∞ are defined by

$$\int_0^a f(x)d_qx = (1-q)a\sum_0^{\infty} f(aq^n)q^n,$$

$$\int_0^{\infty} f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n.$$

Preliminaries

The third Jackson q-Bessel function J_ν (also called Hahn-Exton q-Bessel functions) is defined by the power series

$$J_\nu(x;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n (q;q)_n} x^{2n},$$

and has the normalized form

$$j_\nu(x;q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n (q;q)_n} x^{2n}.$$

It satisfies the following estimate

$$|j_\nu(q^n, q^2)| \leq \frac{(-q^2;q^2)_\infty (-q^{2\nu+2};q^2)_\infty}{(q^{2\nu+2};q^2)_\infty} \begin{cases} 1 & \text{if} \quad n \geq 0 \\ q^{n^2-(2\nu+1)n} & \text{if} \quad n < 0 \end{cases}$$

Preliminaries

The normalized q-Bessel functions satisfies an orthogonality relation

$$c_{q,\nu}^2 \int_0^\infty j_\nu(q^n x, q^2) j_\nu(q^m x, q^2) x^{2\nu+1} d_q x = \frac{q^{-2n(\nu+1)}}{1-q} \delta_{nm}$$

where

$$c_{q,\nu} = \frac{1}{1-q} \frac{(q^{2\nu+2}; q^2)_\infty}{(q^2; q^2)_\infty}.$$

Preliminaries

The function $x \mapsto j_\nu(\lambda x, q^2)$ is a solution of the following q-differential equation

$$\Delta_{q,\nu} f = -\lambda^2 f,$$

where $\Delta_{q,\nu}$ is the q-Bessel operator

$$\Delta_{q,\nu} f(x) = \frac{1}{x^2}[f(q^{-1}x) - (1 + q^{2\nu})f(x) + q^{2\nu} f(qx)].$$

Preliminaries

We denote by

- For $1 \leq p < \infty$ we denote by $\mathcal{L}_{q,p,\nu}$ the set of all functions on $\mathbb{R}_q^+$ for which

$$\|f\|_{q,p,\nu} = [\int_0^{\infty} |f(x)|x^{2\nu+1} d_q x]^{\frac{1}{p}} < \infty.$$

- $d\mu_{\nu,q}(x) = x^{2\nu+1} d_q x.$

Preliminaries

Definition

For $f \in \mathcal{L}_{q,1,\nu}$, the q-Bessel Fourier transform $\mathcal{F}_{q,\nu}$

$$\mathcal{F}_{q,\nu} f(x) = c_{q,\nu} \int_0^\infty f(t) j_\nu(xt, q^2) d\mu_{\nu,q}(t),$$

Definition

The q-Bessel translation operator is defined as follows

$$T^\nu_{q,x} f(y) = c_{q,\nu} \int_0^\infty \mathcal{F}_{q,\nu}(f)(t) j_\nu(xt, q^2) j_\nu(yt, q^2) d\mu_{\nu,q}(t).$$

Definition

The q-translation operator is positive if

$$T^\nu_{q,x} f \geq 0, \quad \forall f \geq 0, \quad \forall x \in \mathbb{R}_q^+.$$

Preliminaries

The domaine of positivity of the q-translation operator is

$$Q_\nu = \{q \in]0,1[, \quad T^\nu_{q,x} \quad \textit{is positive for all } x \in \mathbb{R}^+_q\}.$$

In [5] it was proved that if $-1 < \nu < \nu'$ then $Q_\nu \subset Q_{\nu'}$. As a consequence:

- if $0 \leq \nu$ then $Q_\nu =]0,1[$.
- if $-\frac{1}{2} \leq \nu < 0$ then $]0, q_0[\subset Q_{-\frac{1}{2}} \subset Q_\nu \subseteq]0,1[, \quad q_0 \simeq 0.43$.
- if $-1 \leq \nu < -\frac{1}{2}$ then $Q_\nu \subset Q_{-\frac{1}{2}}$.

Preliminaries

In the rest of this talk we always assume that the q-translation operator is positive.

As a direct consequence of the positivity of the q-translation operator [3] we get

$$|j_\nu(x, q^2)| \leq 1, \quad \forall x \in \mathbb{R}_q^+.$$

Preliminaries

Proposition:

There exists $\alpha, \beta, \eta > 0$ such that:

1. $$\alpha \leq \mid j_\nu(t, q^2) - 1 \mid, \quad \forall t > 1, \quad t \in \mathbb{R}_q^+. \tag{1}$$

2. $$\mid j_\nu(t, q^2) - 1 \mid \leq \beta t^2, \quad \forall t \leq 1, \quad t \in \mathbb{R}_q^+. \tag{2}$$

3. $$\mid j_\nu(t, q^2) - 1 \mid \geq \eta t^2, \quad \forall t \leq 1, \quad t \in \mathbb{R}_q^+. \tag{3}$$

Preliminaries

Theorem:

The q-Bessel Fourier transform satisfies

1. For all functions $f \in \mathcal{L}_{q,p,\nu}$, $p \geq 1$, we have

$$\mathcal{F}^2_{q,\nu} f(x) = f(x), \quad \forall x \in \mathbb{R}^+_q.$$

2. For all functions $f \in \mathcal{L}_{q,2,\nu}$,

$$\| \mathcal{F}_{q,\nu} f \|_{q,2,\nu} = \| f \|_{q,2,\nu} .$$

Corollary:

For any function $f \in \mathcal{L}_{q,2,\nu}$ we have

$$\mathcal{F}_{q,\nu}(T^{\nu}_{q,x} f)(\lambda) = j_\nu(\lambda x, q^2)\mathcal{F}_{q,\nu}(f)(\lambda), \quad \forall \lambda, x \in \mathbb{R}^+_q.$$

Preliminaries

Now we define the finite differences of order $m \in \mathbb{N}$ and step $h \in \mathbb{R}_q^+$ by

$$\Lambda_{q,h}^{m} = \left(T_{q,h}^{\nu} - I\right)^{m}$$

Where I is the unit operator.

Preliminaries

Remark:

For all $m \in \mathbb{N}$ and $h \in \mathbb{R}_q^+$, we have

$$\left(\Lambda_{q,h}^m f\right)(x) = \sum_{0 \leq i \leq m} (-1)^{m-i} \begin{pmatrix} m \\ i \end{pmatrix} \left(T_{q,h}^{\nu}\right)^i f(x).$$

Preliminaries

Lemma:

For all $m \in \mathbb{N}$ and $h \in \mathbb{R}_q^+$, we have

$$\mathcal{F}_{q,\nu}\left(\Lambda_{q,h}^{m} f\right)(x) = \left(j_\nu\left(\lambda h, q^2\right) - 1\right)^m \mathcal{F}_{q,\nu}(f)(x). \tag{4}$$

Auxiliary results

In this section, we will show two important lemmas that we used in the proof of our main results.
Let g be a non-negative, measurable function defined on $\mathbb{R}_q^+$.

Lemma:

Let $m \in \mathbb{N}$. Then

i. If $0 < \alpha \leq m$ is given number, $x^m g(x) \in \mathcal{L}_{q,1,\nu}$ and

$$\int_0^y x^m g(x) d\mu_{\nu,q}(x) = O(y^{m-\alpha}) \textit{ for all } y \in \mathbb{R}_q^+ \tag{5}$$

then $g.\chi_{(y,+\infty)} \in \mathcal{L}_{q,1,\nu}$, where $\chi_{(y,+\infty)}(.)$ is the characteristic function of $(y,+\infty)$, and

$$\int_0^y g(x) d\mu_{\nu,q}(x) = O(y^{-\alpha}) \textit{ for all } y \in \mathbb{R}_q^+ \tag{6}$$

ii. Conversely, if $0 \leq \alpha < m$ and (6) holds, then (5) also holds.

Auxiliary results

Lemma:

Let $m \in \mathbb{N}$. Then

1. If $0 < \alpha \leq m$ is given number, $x^m g(x) \in \mathcal{L}_{q,1,\nu}$ and

$$\int_0^y x^m g(x) d\mu_{\nu,q}(x) = o(y^{m-\alpha}) \text{ as } y \to +\infty \tag{7}$$

   then $g.\chi_{(y,+\infty)} \in \mathcal{L}_{q,1,\nu}$ for large y, where $\chi_{(y,+\infty)}(.)$ is the characteristic function of $(y, +\infty)$, and

$$\int_0^{+\infty} g(x) d\mu_{\nu,q}(x) = o(y^{-\alpha}) \text{ as } y \to +\infty \tag{8}$$

2. Conversely, if $0 < \alpha \leq m$ and both conditions (5) and (8) are hold, then (7) also holds.

Main results

Before giving our main results, we define, first, the generalized Lipschitz classes $H_{q,\alpha}^{m}$, and $h_{q,\alpha}^{m}$.

Definition:

A function f is said belong to $H_{q,\alpha}^{m}$ for $\alpha > 0$ if

$$\left|\Lambda_{q,h}^{m} f(x)\right| = O\left(h^{\alpha}\right), \; h \in \mathbb{R}_{q}^{+}, \tag{9}$$

and is said to belong to $h_{q,\alpha}^{m}$ for $\alpha > 0$ if

$$\left|\Lambda_{q,h}^{m} f(x)\right| = o\left(h^{\alpha}\right) \text{ as } h \to 0. \tag{10}$$

The spaces $H_{q,\alpha}^{1}$ for $\alpha > 0$ and $h_{q,\alpha}^{1}$ for $\alpha > 0$, are called respectively the Lipschitz class $Lip(\alpha)$ and little Lipschtiz class $lip(\alpha)$. The spaces $H_{q,\alpha}^{2}$ for $\alpha > 0$ and $h_{q,\alpha}^{2}$ for $\alpha > 0$, are called respectively the Zygmund class $Zyg(\alpha)$ and little Zygmund class $zyg(\alpha)$.

Theorem:

Let f be a function, $m \in \mathbb{N}$ and $0 < \alpha \leq 2m$.

1. Suppose that $f \in \mathcal{L}_{q,1,\nu}$. If

$$\int_0^y x^{2m} |\mathcal{F}_{q,\nu}(f)(x)| d\mu_{\nu,q}(x) = O(y^{2m-\alpha}), \text{ for all } y \in \mathbb{R}_q^+, \tag{11}$$

then $\mathcal{F}_{q,\nu}(f) \in \mathcal{L}_{q,1,\nu}$ and $f \in H^m_{q,\alpha}$.

2. Conversely, suppose f, $\mathcal{F}_{q,\nu}(f) \in \mathcal{L}_{q,1,\nu}$. If $f \in H^m_{q,\alpha}$ and $\mathcal{F}_{q,\nu}(f)$ is non-negative or non-positive, then (11) holds.

Theorem:

Both statements in the previous theorem remain valid if the right-hand side in (11) is replaced by

$$o\left(y^{2m-\alpha}\right) \text{ as } y \to +\infty, \tag{12}$$

and $f \in H^m_{q,\alpha}$ is replaced by $f \in h^m_{q,\alpha}$, provided that $0 < \alpha < 2m$.

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Thank you for your attention