Uncertainty Principles

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for The Quaternion Linear Canonical Transform

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13th ISAAC Congress 2021 .

3 août 2021

Outline

Quaternion Linear Canonical Transform

Quaternion Linear Canonical Transform

A quaternion $q \in \mathcal{H}$ can be written in this form

$$q = q_0 + \underline{q} = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{ij}q_3$$

where **i** and **j**, satisfy Hamilton's multiplication rules

$$\mathbf{i}^2 = \mathbf{j}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji}.$$

So the modulus of a quaternion q defined by

$$|q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}.$$

It is not difficult to see that

$$|pq| = |p||q|. \ \ \forall p, \ \ q \in \mathcal{H}.$$

Quaternion Linear Canonical Transform

In this paper, we study the quaternion-valued signal $f : \mathbb{R}^2 \to \mathcal{H}$ that can be expressed as

$$f(\underline{x}) = f_0(\underline{x}) + \mathbf{i}f_1(\underline{x}) + \mathbf{j}f_2(\underline{x}) + \mathbf{ij}f_3(\underline{x})$$

where $\underline{x} = \mathbf{i}x_1 + \mathbf{j}x_2 \in \mathbb{R}^2$ and $f_0,\ f_1,\ f_2$ and f_3 are real-valued functions.

Quaternion Linear Canonical Transform

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For $1 \leq r < \infty$, the quaternion modules $L^r(\mathbb{R}^2, \mathcal{H})$ are defined as

$$L^r = L^r(\mathbb{R}^2, \mathcal{H}) = \{f/f : \mathbb{R}^2 \to \mathcal{H},\ \|f\|_{L^r}^r = \int_{\mathbb{R}^2} |f(\underline{x})|^r d\underline{x} < \infty\}.$$

Quaternion Linear Canonical Transform

The inner product of $f(\underline{x})$, $g(\underline{x}) \in L^2(\mathbb{R}^2, \mathcal{H})$ is defined by

$$\langle f(\underline{x}), g(\underline{x})\rangle = \int_{\mathbb{R}^2} f(\underline{x})\overline{g(\underline{x})}d\underline{x}$$

Clearly, $\|f\|_2^2 = \langle f, f\rangle$. Now, we define a norm of $\mathcal{F}(f)$ as

$$|\mathcal{F}(f)(\underline{x})|_Q = \Big(|\mathcal{F}(f_0)(\underline{x})|^2 + |\mathcal{F}(f_1)(\underline{x})|^2 + |\mathcal{F}(f_2)(\underline{x})|^2 + |\mathcal{F}(f_3)(\underline{x}$$

Furthermore, we obtain the $L^r(\mathbb{R}^2, \mathcal{H})$-norm

$$\|\mathcal{F}(f)\|_{Q,r} = \left(\int_{\mathbb{R}^2} |\mathcal{F}(f)(\underline{x})|_Q^r d\underline{x}\right)^{1/r}.$$

Quaternion Linear Canonical Transform

Let $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathbb{R}^{2\times 2}$ be a real matrix parameter such that $\det(A_i) = 1$, for $i = 1, 2$. The (right-sided) QLCT of $f \in L^1(\mathbb{R}^2, \mathcal{H})$ is defined by

$$\mathcal{L}_{A_1,A_2}(f)(\underline{\xi}) = \int_{\mathbb{R}^2} f(x) K^{\mathbf{i}}_{A_1}(x_1, \xi_1) K^{\mathbf{j}}_{A_2}(x_2, \xi_2) d\underline{x}$$

Quaternion Linear Canonical Transform

where the kernel functions of the QLCT above are given by

$$K^{\mathbf{i}}_{A_1}(x_1,\xi_1)=\begin{cases}\frac{1}{\sqrt{2\pi b_1}}e^{(\mathbf{i}/2)((a_1/b_1)x_1^2-(2/b_1)x_1\xi_1+(d_1/b_1)\xi_1^2)-\pi/2},\\ \sqrt{d_1}e^{i(c_1d_1/2)\xi_1^2},\end{cases} \tag{1}$$

$$K^{\mathbf{j}}_{A_2}(x_2,\xi_2)=\begin{cases}\frac{1}{\sqrt{2\pi b_2}}e^{(\mathbf{j}/2)((a_2/b_2)x_2^2-(2/b_2)x_2\xi_2+(d_2/b_2)\xi_2^2)-\pi/2},\\ \sqrt{d_2}e^{\mathbf{j}(c_2d_2/2)\xi_2^2},\end{cases}. \tag{2}$$

Quaternion Linear Canonical Transform

Then from the definition of the QLCT, we can easily see that when $b_1 b_2 = 0$ and $b_1 = b_2 = 0$, the QLCT of a signal is essentially a quaternion chirp multiplication. Therefore, in this work, we always assume $b_1 b_2 \neq 0$, if $b_i \neq 0$ for $i = 1,\ 2$, then $\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}^{-1} = \begin{pmatrix} d_i & -b_i \\ -c_i & a_i \end{pmatrix}$.

(Riemann-Lebesgue lemma)

Theorem

Suppose that $f \in L^1(\mathbb{R}^2, \mathcal{H})$. Then

$$\lim_{|\omega_1| \to \infty} |\mathcal{L}_{A_1,A_2}(f)(\underline{\omega})| = 0, \lim_{|\omega_2| \to \infty} |\mathcal{L}_{A_1,A_2}(f)(\underline{\omega})| = 0.$$

(Inversion formula)

Definition

The (right-sided) inverse QLCT of $g \in L^1(\mathbb{R}^2, \mathcal{H})$

$$\mathcal{L}^{-1}_{A_1,A_2}(g)(\underline{\xi}) = \int_{\mathbb{R}^2} g(\underline{x}) K^{\boldsymbol{j}}_{A_2^{-1}}(x_2, \xi_2) K^{\boldsymbol{i}}_{A_1^{-1}}(x_1, \xi_1) d\underline{x} \qquad (3)$$

(Plancherel theorem of QLCT)

Theorem

Let $f \in L^2(\mathbb{R}^2, \mathcal{H})$ *then*

$$\|\mathcal{L}_{A_1,A_2}(f)\|_{Q,2} = \|f\|_2. \quad (4)$$

(Hausdorff-Young inequality)

Theorem

if $1 \leq r \leq 2$ *and letting* r' *be such that* $1/r + 1/r' = 1$ *then for all* $f \in L^r(\mathbb{R}^2, \mathcal{H})$ *it holds that*

$$\|\mathcal{L}_{A_1,A_2}(f)\|_{Q,r'} \leq \frac{|b_1 b_2|^{-1/2+1/r'}}{2\pi} \|f\|_r. \tag{5}$$

In this section, based on the techniques of Donoho-Stark [5] and K.I Kou, we will show uncertainty principle of concentration-type for Quaternion Linear Canonical transform (QLCT).

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

Definition

We consider a pair of orthogonal projections on $L^2(\mathbb{R}^2, \mathcal{H})$. The first is P_S operator defined by :

$$P_S f = \chi_S f, \tag{6}$$

and the second is Q_Σ operator defined by :

$$Q_\Sigma f = \mathcal{L}^{-1}_{A_1,A_2}\left[\chi_\Sigma \mathcal{L}_{A_1,A_2}(f)\right], \tag{7}$$

where S and Σ are two measurable subsets of $\mathbb{R}^2$, and χ_S denote the characteristic function of S.

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

Definition

Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$ and let $f \in L^2(\mathbb{R}^2, \mathcal{H})$ be a nonzero function. We say that f is ε_S-concentrated on S if :

$$\|P_{S^c} f\|_2 \leq \varepsilon_S \|f\|_2. \tag{8}$$

Similarly we say that f is ε_Σ-concentrated on Σ for the Quaternion Linear Canonical transform if

$$\|Q_{\Sigma^c} f\|_2 \leq \varepsilon_\Sigma \|f\|_2. \tag{9}$$

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

We define the norm of P_S as following

$$\|P_S\| = \sup_{f \in L^2(\mathbb{R}^2, \mathcal{H})} \frac{\|P_S(f)\|_2}{\|f\|_2}.$$

In the same way, the norm of Q_Σ is defined by

$$\|Q_\Sigma\| = \sup_{f \in L^2(\mathbb{R}^2, \mathcal{H})} \frac{\|Q_\Sigma(f)\|_2}{\|f\|_2}.$$

Since P_S and Q_Σ are projections, it is clear that $\|P_S\| = \|Q_\Sigma\| = 1$.
If $|\Sigma| < \infty$ where Σ is a set of finite measure of $\mathbb{R}^2$, we have

$$|\Sigma| = \int_\Sigma d\underline{x}.$$

In the following proposition we give an other inequality proved by Donoho-Stark [5] in the one dimensional case. We may derive the analogue result to quaternion-valued signals. We consider unit energy signal for simplification. Then by Plancherel's theorem we get $\|f\|_2 = \|\mathcal{L}_{A_1,A_2}(f)\|_{Q,2} = 1$.

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

Let S and Σ be two measurable sets of $\mathbb{R}^2$ and assume that $\varepsilon_S + \varepsilon_\Sigma < 1$, f is ε_S-concentrated on S and $\mathcal{L}_{A_1,A_2}(f)$ is ε_Σ-concentrated on Σ. Then

$$\left(\frac{|b_1 b_2|^{-1/2}}{2\pi}\right)^2 |S||\Sigma| \geq (1 - \varepsilon_S - \varepsilon_\Sigma)^2.$$

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

Let $f \in L^2(\mathbb{R}^2, \mathcal{H})$ and $S,\ \Sigma \subset \mathbb{R}^2$ be two measurable sets such that $|S||\Sigma| < \left(\frac{2\pi}{|b_1 b_2|^{-1/2}}\right)^2$ and let $\varepsilon_S, \varepsilon_\Sigma > 0$ such that $\varepsilon_S^2 + \varepsilon_\Sigma^2 < 1$. If f is ε_S-concentrated on S and ε_Σ-concentrated on Σ for the Quaternion Linear Canonical transform, then

$$\left(\frac{|b_1 b_2|^{-1/2}}{2\pi}\right)^2 |S||\Sigma| \geq \left(1 - \sqrt{\varepsilon_S^2 + \varepsilon_\Sigma^2}\right)^2 .$$

$$r(\underline{x}) = \begin{cases} f(\underline{x}) + n(\underline{x}), & x \in S^c\,; \\ 0, & x \in S. \end{cases}$$

Here, without loss of generality we can assume that $n = 0$ on S. Equivalently,

$$r = (I - P_S)f + n.$$

We propose to construct f from r. This construction requires the following definition.

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

We say that f can be stably reconstructed from r, if there exists a linear operator K and a constant C such that :

$$\|f - Kr\|_2 \leq C\|n\|_2. \tag{10}$$

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

If S and Σ are two measurable sets of $\mathbb{R}^2$ satisfy the condition $0 < |S||\Sigma| < \left(\frac{2\pi}{|b_1 b_2|^{-1/2}}\right)^2$ then f can be stably reconstructed from r. That is, there exists a constant C such that

$$C \leq \left(1 - \frac{|b_1 b_2|^{-1/2}}{2\pi}\sqrt{|S||\Sigma|}\right)^{-1}.$$

Donoho-Stark's uncertainty principle for Quaternion Linear Canonical transform

Using the similar method given by K. I. Kou, we give an algorithm for computing Kr. Let

$$K = (I - P_S Q_\Sigma)^{-1} = \sum_{k=0}^{\infty} (P_S Q_\Sigma)^k$$

Put

$$f^{(n)} = \sum_{k=0}^{n} (P_S Q_\Sigma)^k r,$$

then

$$\begin{cases} f^{(0)} = r \\ f^{(n+1)} = r + P_S Q_\Sigma f^{(n)} \\ f^{(n)} \to Kr \ \ as \ \ n \to \infty. \end{cases}$$

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In this section we will prove a quantitative uncertainty inequality about the essential supports of a nonzero function $f \in L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r \leq 2$ and its Quaternion Linear Canonical transform.

L^r-uncertainty principles for the Quaternion Linear Canonical transform

Theorem

Let $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r \leq 2$, *then*

$$\|\mathcal{L}_{A_1,A_2}(f)\|_{Q,r'} \leq \frac{|b_1 b_2|^{-1/2}}{2\pi} |\text{supp } \mathcal{L}_{A_1,A_2}(f)|^{1/r'} |\text{supp } f|^{1/r'} \|f\|_r$$

where $r' = \frac{r}{r-1}$.

L^r-uncertainty principles for the Quaternion Linear Canonical transform

Let $f \in L^2(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r \leq 2$, then

$$1 < \frac{|b_1 b_2|^{-1/2+1/r'}}{2\pi} |\text{supp } \mathcal{L}_{A_1,A_2}(f)|^{\frac{r'-2}{2r'}} |\text{supp } f|^{\frac{2-r}{2r}}$$

where $r' = \frac{r}{r-1}$.

L^r-uncertainty principles for the Quaternion Linear Canonical transform

we say that a function $f \in L^r(\mathbb{R}^2, \mathcal{H})$ is ε_S-concentrated to S in $L^r(\mathbb{R}^2, \mathcal{H})$-norm if and only if

$$\|f - P_S f\|_r \leq \varepsilon_S \|f\|_r. \tag{11}$$

$\mathcal{L}_{A_1,A_2}(f)$ is ε_Σ-concentrated on Σ in $L^{r'}(\mathbb{R}^2, \mathcal{H})$-norm if and only if

$$\|\mathcal{L}_{A_1,A_2}(f) - \mathcal{L}_{A_1,A_2}(Q_\Sigma f)\|_{Q,r'} \leq \varepsilon_\Sigma \|\mathcal{L}_{A_1,A_2}(f)\|_{Q,r'}. \tag{12}$$

L^r-uncertainty principles for the Quaternion Linear Canonical transform

Theorem

Let $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r \leq 2$ and let S, Σ be two measurable subsets of $\mathbb{R}^2$. If f is ε_S-concentration to S in $L^r(\mathbb{R}^2, \mathcal{H})$-norm and $\mathcal{L}_{A_1,A_2}(f)$ is ε_Σ-concentration to Σ in $L^{r'}(\mathbb{R}^2, \mathcal{H})$-norm, then

$$\|\mathcal{L}_{A_1,A_2}(f)\|_{Q,r'} \leq \frac{|b_1 b_2|^{-1/2}}{2\pi} \left(\frac{|S|^{\frac{1}{r'}} |\Sigma|^{\frac{1}{r'}} + \varepsilon_S |b_1 b_2|^{1/r'}}{1 - \varepsilon_\Sigma} \right) \|f\|_r.$$

We are now in position to give another principle of concentration type for the Quaternion Linear Canonical transform in $L^1(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$ proved by Donoho-Strak [5] in the one dimensional case, this principle is given by the following theorem.

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Theorem

Let S and Σ be two measurable subsets of $\mathbb{R}^2$ and $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r \leq 2$. If f is ε_S-concentration to S in $L^1(\mathbb{R}^2, \mathcal{H})$-norm and $\mathcal{L}_{A_1,A_2}(f)$ is ε_Σ-concentration to Σ in $L^{r'}(\mathbb{R}^2, \mathcal{H})$-norm, then

$$\|\mathcal{L}_{A_1,A_2}(f)\|_{Q,r'} \leq \frac{|b_1 b_2|^{-1/2}}{2\pi} \frac{|S|^{\frac{1}{r'}} |\Sigma|^{\frac{1}{r'}}}{(1-\varepsilon_S)(1-\varepsilon_\Sigma)} \|f\|_r$$

where $r' = \dfrac{r}{r-1}$.

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Let S and Σ be two measurable subsets of $\mathbb{R}^2$, and $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^2(\mathbb{R}^2, \mathcal{H})$ if f is ε_S concentrated to S in $L^1(\mathbb{R}^2, \mathcal{H})$-norm and $\mathcal{L}_{A_1,A_2}(f)$ is ε_Σ concentrated to Σ in $L^2(\mathbb{R}^2, \mathcal{H})$-norm, then

$$(1-\varepsilon_S)(1-\varepsilon_\Sigma) \leq \frac{|b_1 b_2|^{-1/2}}{2\pi}\sqrt{|S||\Sigma|}.$$

Let $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^{r_1}(\mathbb{R}^2, \mathcal{H}) \cap L^{r_2}_2(\mathbb{R}^2, \mathcal{H})$, $1 < r_1 < r_2 < 2$ and let S and Σ be two measurable subsets of $\mathbb{R}^2$. If f is ε_S concentrated to S in $L^{r_1}(\mathbb{R}^2, \mathcal{H})$-norm and $\mathcal{L}_{A_1,A_2}(f)$ is ε_Σ concentrated to Σ in $L^{r'_2}(\mathbb{R}^2, \mathcal{H})$-norm, $r'_2 = \frac{r_2}{r_2-1}$ then

$$\|\mathcal{L}_{A_1,A_2}\|_{Q,r'_2} \leq \frac{|b_1 b_2|^{-1/2+1/r'_1}}{2\pi} \frac{|S|^{\frac{r_2-r_1}{r_1 r_2}} |\Sigma|^{\frac{r'_1-r'_2}{r'_1 r'_2}}}{(1-\varepsilon_S)(1-\varepsilon_\Sigma)} \|f\|_{r_2},$$

where $r'_1 = \frac{r_1}{r_1-1}$.

Let $B^r(\Sigma)$, $1 \leq r \leq 2$, be the set of functions $g \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$ that are bandlimited to Σ i.e ($g \in B^r(\Sigma)$ implies $Q_\Sigma g = g$).
We say that f is ε_Σ-bandlimited to Σ in $L^r(\mathbb{R}^2, \mathcal{H})$-norm if there is a $g \in B^r(\Sigma)$ with

$$\|f - g\|_r \leq \varepsilon_\Sigma \|f\|_r.$$

The space $B^r(\Sigma)$ satisfies the following property.

Bandlimited principle-type

Let S and Σ be two measurable subsets of $\mathbb{R}^2$. For $g \in B^r(\Sigma)$, $1 \leq r \leq 2$,

$$\|P_S g\|_r \leq \frac{|b_1 b_2|^{-1/2+1/r'}}{2\pi} |S|^{\frac{1}{r}} |\Sigma|^{\frac{1}{r}} \|g\|_r$$

Bandlimited principle-type

Let S and Σ be two measurable subsets of $\mathbb{R}^2$, and $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^{r_1}(\mathbb{R}^2, \mathcal{H}) \cap L_2^{r_2}(\mathbb{R}^2, \mathcal{H})$, $1 < r_1 < r_2 < 2$. If f is ε_S concentrated to S in $L^{r_1}(\mathbb{R}^2, \mathcal{H})$-norm and ε_Σ-bandlimited to Σ in $L^{r_2}(\mathbb{R}^2, \mathcal{H})$-norm, $r_2' = \frac{r_2}{r_2-1}$ then

$$\|f\|_{r_1} \leq \frac{|S|^{\frac{r_2-r_1}{r_1 r_2}}}{1-\varepsilon_S}\left[(1+\varepsilon_\Sigma)\frac{|b_1 b_2|^{-1/2+1/r_2'}}{2\pi}|S|^{\frac{1}{r_2}}|\Sigma|^{\frac{1}{r_2}} + \varepsilon_\Sigma\right]\|f\|_{r_2}$$

where $r_1' = \frac{r_1}{r_1-1}$.

Bandlimited principle-type

Let S and Σ be two measurable subsets of $\mathbb{R}^2$ and $f \in L^1(\mathbb{R}^2, \mathcal{H}) \cap L^r(\mathbb{R}^2, \mathcal{H})$, $1 \leq r \leq 2$. If f is ε_S-concentrated to S and f is ε_Σ-bandlimited to Σ in $L^r(\mathbb{R}^2, \mathcal{H})$-norm then

$$\frac{1 - \varepsilon_S - \varepsilon_\Sigma}{1 + \varepsilon_\Sigma} \leq \frac{|b_1 b_2|^{-1/2+1/r'}}{2\pi} |S|^{\frac{1}{r}} |\Sigma|^{\frac{1}{r}}.$$

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Thank you for your attention