On a Non-local Problem for The Loaded Parabolic-hyperbolic Type Equation with Non-Linear Terms

Obidjon Abdullaev

obidjon.mth@gmail.com

V.I.Romanovskiy Institute of Mathematics , Tashkent, Uzbekistan

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Note, that with intensive research on problems of optimal control of the agro-economical system, regulating the label of ground waters and soil moisture, it has become necessary to investigate BVPs for a loaded partial differential equations.

Integral boundary conditions have various applications in thermo-elasticity, chemical engineering, population dynamics, etc. In this work we consider parabolic-hyperbolic type equation fractional order involving non-linear loaded term:

$$0 = \begin{cases} u_{xx} - {}_C D_{ot}^{\alpha} u + a_1(x,\, t)\, u^{p_1}(x,\, t) + f_1(x,\, t;\, u(x, 0)), \text{ for } t > 0, \\ u_{xx} - u_{tt} + a_2(x,\, t)\, u^{p_2}(x,\, t) + f_2(x,\, t;\, u(x,\, 0)), \text{ for } t < 0, \end{cases} \tag{1}$$

where

$${}_C D_{0y}^{\alpha} f(y) = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} f'(t)dt, \quad 0 < \alpha < 1, \tag{2}$$

is Caputo differential operator, $a_i(x,\, t)$, $f_i(x,\, t;\, u\,(x,\, 0))$ are given functions, and $p_i = const > 0$, $0 < \alpha < 1$, $i = 1,\, 2$.

Let $J = \{(x, t); t = 0, 0 < x < l\}$, $\Omega_1 = \{(x, t) : 0 < x < l, 0 < t < h\}$, $\Omega_2 = \{(x, t) : 0 < x + t < l, 0 < x - t < l,\ t < 0\}$ and $\Omega = \Omega_1 \cup J \cup \Omega_2$

Problem *G***.** It is required to find a solution $u(x,\ t)$ of the equation (1) with the following properties:
1) $u(x,t) \in C(\overline{\Omega}) \cap C^2(\Omega_2)$, $u_{xx}, {}_C D_{ot}^{\alpha} u \in C(\Omega_1)$, $u_x \in C^1\left(\overline{\Omega}_1 \setminus t = h\right)$;
2) $u(x,t)$ satisfy boundary value conditions:

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = \varphi_1(t),\ \beta_1 u(l,t) + \beta_2 u_x(l,t) = \varphi_2(t),\ 0 \le t < h; \tag{3}$$

$$\gamma_1 u\left(\frac{x}{2}, -\frac{x}{2}\right) + \gamma_2 u\left(\frac{x+l}{2}, \frac{x-l}{2}\right) = \psi(x),\ 0 \le x \le l; \tag{4}$$

and integral gluing condition:

$$\lim_{t\to+0} t^{1-\alpha} u_t(x,\ t) = \lambda_1(x)\, u_t(x,\ -0) + \lambda_2(x)\, u_x(x,\ -0)$$

$$+\lambda_3(x) u(x,\ 0) + \lambda_4(x),\ 0 < x < 1, \tag{5}$$

where $\psi(x)$, $\varphi_i(t)$, $\lambda_k(x)\,(k = \overline{1,\ 4})$ are given continuous functions and $\alpha_i, \beta_i, \gamma_i,\ (i = 1,2)$ are given constants, such that $\sum\limits_{k=1}^{3} \lambda_k^2(x) \neq 0$ and $\alpha_1^2 + \alpha_2^2 \neq 0$, $\beta_1^2 + \beta_2^2 \neq 0$, $\gamma_1^2 + \gamma_2^2 \neq 0$.

Remarks

- **Remark.1.** As we known, the same above problems for the equation (1) at $\alpha = 1$ have not been investigated, too. On the another hand, we would like to note, that fundamental solution of equation (1) for $t > 0$ at $\alpha = 1$, completely coincides with the fundamental solution of the heat equation $u_{xx} - u_t = 0$. Therefore, all results in this work remain valid in case $\alpha = 1$, too.
- **Remark.2.** If $\alpha_1 = \beta_1 = \gamma_1 = 0$ or $\alpha_1 = \beta_1 = \gamma_2 = 0$, then the Problem G becomes a local boundary value problem (BVP) with the secondary boundary conditions on the parabolic domain, moreover investigation of the new local problem will be reduced to the Volterra type non-linear integral equations.
- **Remark.3.** If $\alpha_2 = \beta_2 = 0$ and $\gamma_1, \gamma_2 \neq 0$, then the Problem G becomes a non-local BVP with the first boundary conditions on the parabolic domain, however investigation of the new non-local problem will be reduced to the Fredholm type non-linear integral equations.

We note that solution of the non-local problem with condition (4) and $u(x,0)=\tau(x)$ for equation (1) in Ω_2 at $\gamma_1\neq\gamma_2$, has a form:

$$u(x,y)=\frac{\gamma_1\tau(x+y)-\gamma_2\tau(x-y)}{\gamma_1-\gamma_2}+\frac{\gamma_1\psi(x+y)-\gamma_2\psi(x-y)}{\gamma_1-\gamma_2}-$$

$$-\frac{1}{4}\int\limits_{x+y}^{x-y}d\eta\int\limits_{x+y}^{0}a_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)u^{p_2}\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2}\right)d\xi$$

$$-\frac{1}{4}\int\limits_{x+y}^{x-y}d\eta\int\limits_{x+y}^{0}f_2\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2};\tau\left(\frac{\xi+\eta}{2}\right)\right)d\xi. \quad (6)$$

Further, from the equation (1) as $y\to+0$ taking into account (2), (5) and

$$\lim_{y\to 0}D_{0y}^{\alpha-1}f(y)=\Gamma(\alpha)\lim_{y\to 0}y^{1-\alpha}f(y),$$

we derive

$$\tau''(x)-\Gamma(\alpha)\lambda_1(x)\,u_y(x,\,-0)-\Gamma(\alpha)\lambda_2(x)\,\tau'(x)$$

$$-\Gamma(\alpha)\lambda_3(x)\tau(x)-\Gamma(\alpha)\lambda_4(x)+a_1(x,0)\tau^{p_1}(x)+f_1(x,0;\tau(x))=0,\ 0<x<1.$$

Hence, taking into account (see (6))

$$u_y(x,-0)=\frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\tau'(x)-\frac{2}{\gamma_1-\gamma_2}\psi'(x)$$

$$+\frac{1}{2}\int\limits_0^x a_2\left(\frac{\xi+x}{2},\frac{\xi-x}{2}\right)u^{p_2}\left(\frac{\xi+x}{2},\frac{\xi-x}{2}\right)d\xi$$

$$+\frac{1}{2}\int\limits_0^x f_2\left(\frac{\xi+x}{2},\,\frac{\xi-x}{2};\,\tau\left(\frac{\xi+x}{2}\right)\right)d\xi,$$

we obtain

$$\tau''(x)-\Gamma(\alpha)\left(\frac{\gamma_1+\gamma_2}{\gamma_1-\gamma_2}\lambda_1(x)+\lambda_2(x)\right)\tau'(x)-\Gamma(\alpha)\lambda_3(x)\,\tau(x)$$

$$-\frac{\Gamma(\alpha)}{2}\lambda_1(x)\int\limits_0^x f_2\left(\frac{\xi+x}{2},\frac{\xi-x}{2};\tau\left(\frac{\xi+x}{2}\right)\right)\,d\xi$$

$$+a_1\,(x,0)\tau^{p_1}(x)+f_1\,(x,0;\tau(x))$$

$$=\frac{\Gamma(\alpha)}{2}\lambda_1(x)\int\limits_0^x a_2\left(\frac{\xi+x}{2},\frac{\xi-x}{2}\right)u^{p_2}\left(\frac{\xi+x}{2},\frac{\xi-x}{2}\right)\,d\xi$$

$$+\Gamma(\alpha)\lambda_4(x)+\frac{2\Gamma(\alpha)}{\gamma_1-\gamma_2}\lambda_1(x)\psi'(x),\quad 0<x<1. \tag{7}$$

From the class for solution of the problem G and using by the boundary conditions (3) and (4), we obtain

$$\alpha_1\tau(0)+\alpha_2\tau'(0)=\varphi_1(0), \tag{8}$$

$$\beta_1\tau(l)+\beta_2\tau'(l)=\varphi_2(0), \tag{9}$$

$$\gamma_1^2\tau(0)-\gamma_2^2\tau(l)=_1\psi(0)-\gamma_2\psi(l). \tag{10}$$

We assume, that $\alpha_1 = 0$ but $\gamma_2 \neq 0$ or $\beta_1 = 0$ but $\gamma_1 \neq 0$, (see **Remark.2.**), then based on the (8), (9) and (10), we can formulate next remark:

Remark.4. If $\alpha_1 = \beta_2 = 0$, $\gamma_1 \neq 0$ or $\alpha_2 = \beta_1 = 0$, $\gamma_2 \neq 0$ then the Problem G becomes a non-local BVPs, investigations which will be reduced to the Volterra type non-linear integral equations.

By integration the equation (7) with initial value conditions $\tau(0) = A$ and $\tau'(0) = B$ we derive

$$\tau(x) - \Gamma(\alpha)\int_0^x K_1(x,t)\tau(t)dt + \int_0^x (x-t)a_1(t,0)\tau^{p_1}(t)dt$$

$$-\frac{\Gamma(\alpha)}{2}\int_0^x (x-t)\lambda_1(t)dt\int_0^t f_2\left(\frac{\xi+t}{2},\frac{\xi-t}{2};\tau\left(\frac{\xi+t}{2}\right)\right)d\xi$$

$$+\int_0^x (x-t)f_1(t,0;\tau(t))dt$$

$$=\frac{\Gamma(\alpha)}{2}\int_0^x (x-t)\lambda_1(t)dt\int_0^t a_2\left(\frac{\xi+t}{2},\frac{\xi-t}{2}\right)u^{p_2}\left(\frac{\xi+t}{2},\frac{\xi-t}{2}\right)d\xi+g_1(x), \quad (11)$$

where

$$K_1(x,t)=\left[(x-t)\big(\lambda_1(t)+\lambda_2(t)\big)\right]'-(x-t)\lambda_3(t);$$

$$g_1(x)=\Gamma(\alpha)\int\limits_0^x(x-t)\lambda_4(t)dt-\Gamma(\alpha)\int\limits_0^x(x-t)\lambda_1(t)\psi'(t)dt+\varphi_2(0)x+\psi(0).$$

Theorem. We suppose that $p_i = \mathrm{const} > 1$ and the following conditions are fulfilled:

$$a_i(x,y) \in C\left(\overline{\Omega}_i\right) \cup C^1\left(\Omega_i\right),\ f_i(x,\ y,\ u(x,0)) \in C\left(\overline{\Omega}_i\right) \cup C^1\left(\Omega_i\right); \quad (12)$$

$$\varphi_i(y) \in C[0,h] \cup C^1(0,h),\ \psi(x) \in C[0,1] \cup C^2(0,1); \quad (13)$$

$$\lambda_k(x) \in C[0,1] \cap C^1(0,1)\,(k = 1,2,3,4); \quad (14)$$

$$\left|f_i(x,(i-1)y;u_2(x,0)) - f_i(x,(i-1)y;u_1(x,0))\right| \le L_i\left|u_2(x,0) - u_1(x,0)\right|, \quad (15)$$

for all $(x,y) \in \Omega_i$, where $L_i = \mathrm{const} > 0\,(i = 1,2)$. Then the problem has a unique solution.

Volterra type integral equations

By virtue of properties (12) and (13), we have estimates:

$$\left\|K_1(x,t)\right\|_C \le M_1,\ \left\|g_1(x,t)\right\|_C \le g_{10},\ \left\|f_i(x,(i-1)y;u(x,0))\right\|_C \le f_{i0}, \tag{16}$$

where $M_1,\ g_{10},\ f_{i0} = \mathrm{const} > 0\,(i=1,2)$. The equations (6) and (11) we consider as a system of nonlinear integral equations of Volterra type second order with respect to unknown functions $\tau(x)$ and $u(x,y)$ for $y \le 0$ [3]

$$\begin{cases} u(x,y) = \mathbb{S}(x,y,u;\tau), (x,y) \in \Omega_2; \\ \tau(x) = \mathbb{T}(x,y,u;\tau),\quad 0 < x < 1. \end{cases} \tag{17}$$

We define a sequence of functions $\tau_n(x)$ and $u_n(x,\, y)\,(n = 0, 1, ...)$ from the following system of recurrent equations:

$$\begin{cases} u_0(x,\, y) = g_1(x) + \psi\left(\frac{x-y}{2}\right) - \psi\left(\frac{x+y}{2}\right),\ u_n(x,\, y) = \mathbb{S}\left(x,\, y, u_{n-1};\, \tau_{n-1}\right); \\ \tau_0(x) = g_1(x),\ \tau_n(x) = \mathbb{T}\left(x,\, y, u_{n-1};\, \tau_{n-1}\right). \end{cases} \tag{18}$$

By virtue of properties (14),(15) and estimates(16), we have following estimates

$$\left\|a_i(x,\, y)\right\|_C \leq m_i,\ \left\|\lambda_1(x)\right\|_C \leq \lambda_{10},\ \left\|\tau_0(x)\right\|_C \leq c_{11},\ \|u_0(x,\, y)\|_C \leq c_{21},$$

where $m_i,\ \lambda_{10},\ c_{i1} = \mathrm{const} > 0\,(i = 1,\, 2)$.

Further, taking these estimates into account, from the iteration process (18) we derive

$$\begin{cases} \left\|\tau_1(x)-\tau_0(x)\right\|_C \leq \Gamma(\alpha) M_1 c_{11} x + \left(m_1 c_{11}^{p_1} + f_{10}\right) x^2 + \frac{\Gamma(\alpha)}{2} \lambda_{10}\left(f_{20} + m_2 c_{21}^{p_2}\right) \\ \qquad \|u_1(x, y) - u_0(x, y)\|_C \leq \frac{1}{2}\left(m_2 c_{21}^{p_2} + f_{20}\right)|x+y| \leq \frac{\gamma}{2}|x+y|, \end{cases}$$

where

$$\beta = max\left\{\Gamma(\alpha) M_1 c_{11},\ m_1 c_{11}^{p_1} + f_{10},\ \frac{\Gamma(\alpha)}{2}\lambda_{10}\left(f_{20} + m_2 c_{21}^{p_2}\right)\right\}, \gamma = m_2 c_{21}^{p_2} + f_{20}.$$

Continuing the above reasoning for arbitrary n, we have:

$$\left\|\tau_n(x)-\tau_{n-1}(x)\right\|_C \le 4^{n-1}\Gamma^{n-1}(\alpha)\,\lambda_{10}^{n-1}L_2^{n-1}\,\frac{x^n}{n!},\ 0\le x\le 1,$$

$$\left\|u_n(x,\,y)-u_{n-1}(x,\,y)\right\|_C \le 4^{n-2}\Gamma^{n-2}(\alpha)\,\lambda_{10}^{n-2}L_2^{n-2}\,\frac{x^{n-1}}{(n-1)!},\ (x,y)\in\overline{\Omega}_2.$$

By virtue of the obtained estimates, we conclude that the functional sequences of functions $\{\tau_n(x)\}_{n=1}^{\infty}$ and $\{u_n(x,\,y)\}_{n=1}^{\infty}$ has a unique limit functions $\tau\,(x)$ and $u\,(x,\,y)$:

$$\lim_{n\to\infty}\tau_n(x)=\tau\,(x),\quad \lim_{n\to\infty}u_n(x,\,y)=u\,(x,\,y).$$

Thus, the existence of a solution of the system (17) has been proved.

After determination $\tau(x)$ we restore the unique solution of the considering problem B in the domain Ω_2 as a solution of the non-local problem (see the equation (7)). Further, we take the existence of function $\tau(x)$ into account and we will write the solution of second boundary value problem for the equation (1) in domain Ω_1, which has form [1]:

$$u(x, y) = \int_0^y G_\xi(x, y, 0, \eta)\,\varphi_2(\eta)\,d\eta - \int_0^y G_\xi(x, y, 1, \eta)\,\varphi_1(\eta)\,d\eta$$

$$+\int_0^1 G_0(x-\xi, y)\,\tau(\xi)\,d\xi - \int_0^y \int_0^1 G(x, y, \xi, \eta) f_1(\xi, \eta; \tau(\xi))\,d\xi\,d\eta$$

$$-\int_0^y \int_0^1 G(x, y, \xi, \eta) a_1(\xi, \eta)\,u^{p_1}(\xi, \eta)\,d\xi\,d\eta. \tag{19}$$

where

$$G_0(x-\xi,\, y) = \frac{1}{\Gamma(1-\alpha)} \int\limits_0^y (y-\eta)^{-\alpha} G(x,\, \eta,\, \xi,\, 0)\, d\,\eta,$$

$$G(x,\, y,\, \xi,\, \eta) = \frac{(y-\eta)^{\alpha/2-1}}{2} \sum_{n=-\infty}^{\infty} \left[e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\alpha/2}}\right) - e^{1,\alpha/2}_{1,\alpha/2}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\alpha/2}}\right)\right]$$

is Green's function of the second boundary value problem for the equation (1) in the domain Ω_1 [1], [2],

$$e^{1,\,\delta}_{1,\,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\delta-\delta n)}$$

is Wright type function.

It is investigated the last equation as a nonlinear Volterra type integral equation of the second kind

$$u(x,y)+\int_0^y\int_0^1 K_2(\xi,\eta)\,u^{p_1}(\xi,\eta)\,d\xi\,d\eta=F(x,y)$$

by well known methods from the work [5], where

$$F(x,y)=\int_0^y G_\xi(x,y,0,\eta)\,\varphi_2(\eta)\,d\eta-\int_0^y G_\xi(x,y,1,\eta)\,\varphi_1(\eta)\,d\eta$$

$$+\int_0^1 G_0(x-\xi,y)\,\tau(\xi)\,d\xi$$

$$-\int_0^y\int_0^1 G(x,y,\xi,\eta)f_1(\xi,\eta;\tau(\xi))\,d\xi\,d\eta.$$

The Theorem is proved.

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Thank You !