Harmonische Analysis — Blatt 9

Es ist unmöglich, die Schönheit der Naturgesetze angemessen zu vermitteln, wenn jemand die Mathematik nicht versteht. Ich bedaure das, aber das ist wohl so.
(Richard Feynman; 1918–1988)

Problems

9.1. Let $G$ be a compact group. A function $f : G \to \mathbb{C}$ is called central, if $f(xy) = f(yx)$ holds true for all $x, y \in G$. We denote by

$$L^p_{\text{cent}}(G) = \{ f \in L^p(G) \mid f \text{ central} \}$$

the space of central $L^p$-functions and similarly by $C_{\text{cent}}(G)$ the space of continuous central functions.

(a) The spaces $L^p_{\text{cent}}(G)$ and $C_{\text{cent}}(G)$ are commutative Banach*-algebras with respect to the convolution $\ast$. Prove this.

(b) The set of characters $\chi_x$ defined as $\chi_x(x) = \text{tr} \xi(x)$ for irreducible unitary representations $\xi$ of $G$ forms an orthonormal basis of the Hilbert space $L^2_{\text{cent}}(G)$.

(c) Let $A$ be one of the algebras $L^p_{\text{cent}}(G)$ for $1 \leq p < \infty$ or $C_{\text{cent}}(G)$ and define for an irreducible representation $\xi$ of dimension $d_\xi$

$$h_\xi(f) = \frac{1}{d_\xi} \int_G f(x) \overline{\chi_x(x)} \, dx.$$  

Show that $h_\xi \in \text{Hom}(A, \mathbb{C}) \setminus \{0\} = \sigma(A)$ and that there is a bijection between equivalence classes of unitary representations $[\xi]$ and the spectrum $\sigma(A)$.

9.2. We consider the group $SU(2)$ of unitary $2 \times 2$ matrices with determinant equal to 1 and one unitary representation $\pi : SU(2) \to U(d)$ of dimension $d$.

(a) Let $T \subset SU(2)$ be the set of diagonal matrices in $SU(2)$, i.e., the set of matrices of the form

$$R_\theta = \begin{pmatrix} e^{2\pi i \theta} & 0 \\ 0 & e^{-2\pi i \theta} \end{pmatrix}, \quad \theta \in \mathbb{R}$$

Show that there exist numbers $\alpha_k \in \mathbb{N}_0$ such that for all $\theta \in \mathbb{R}$

$$\text{tr} \, \pi(R_\theta) = \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k \theta}.$$  

(*)

(b) Let $\theta_1, \theta_2 \in \mathbb{R}$. Show that there exists a matrix $A \in SU(2)$ with

$$AR_{\theta_1} = R_{\theta_2}A$$

if and only if $\theta_1 + \theta_2 \in \mathbb{Z}$ or $\theta_1 - \theta_2 \in \mathbb{Z}$. Use this to prove that in (*) the relation $\alpha_k = \alpha_{-k}$ holds true for all $k$.

Topics as preparation

9.3. Schrödinger operators with periodic potential are operators of the form

$$A = -\Delta + q(x) : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

where $q : \mathbb{R}^n \to \mathbb{R}$ is continuous and periodic with respect to some lattice $\Lambda \subset \mathbb{R}^n$

$$q(x + y) = q(x) \quad \text{for } x \in \mathbb{R}^n, y \in \Lambda.$$  

Let $\Omega$ be a translation cell of $\Lambda$ and $\Xi = \mathbb{R}^n / \Lambda^\circ$. 
Lemma. The operator $A$ decomposes as direct integral

$$A = \int_{\Xi}^\oplus A(\xi)$$

for operators $A(\xi) : H^2_\xi(\Omega) \to L^2(\Omega)$, where $H^2_\xi(\Omega)$ denotes the subspace of Sobolev functions $f \in H^2(\Omega)$ satisfying $\xi$-quasiperiodic boundary conditions.

Lemma. Each operator $A(\xi)$ has a compact resolvent $(\lambda - A(\xi))^{-1} \in L(L^2(\Omega))$.

Lemma. The spectrum of the operator $A$ (seen as unbounded operator on $L^2(\Omega)$) is purely essential spectrum and satisfies

$$\sigma(A) = \bigcup_{\xi \in \Xi} \sigma_p(A(\xi)).$$