Harmonische Analysis — Blatt 9

Es ist unmöglich, die Schönheit der Naturgesetze angemessen zu vermitteln,
wenn jemand die Mathematik nicht versteht. Ich bedaure das, aber das ist wohl so.
(Richard Feynman; 1918–1988)

Problems

9.1. Let G be a compact group. A function $f : G \to \mathbb{C}$ is called central, if $f(xy) = f(yx)$ holds true for all $x, y \in G$. We denote by

$$\mathrm{L}^p_{\mathrm{cent}}(G) = \{f \in \mathrm{L}^p(G) \mid f \text{ central}\}$$

the space of central L^p-functions and similarly by $\mathrm{C}_{\mathrm{cent}}(G)$ the space of continuous central functions.

(a) The spaces $\mathrm{L}^p_{\mathrm{cent}}(G)$ and $\mathrm{C}_{\mathrm{cent}}(G)$ are commutative Banach-* algebras with respect to the convolution $\star$. Prove this.

(b) The set of characters χ_ξ defined as $\chi_\xi(x) = \operatorname{tr}\xi(x)$ for irreducible unitary represenations ξ of G forms an orthonormal basis of the Hilbert space $\mathrm{L}^2_{\mathrm{cent}}(G)$.

(c) Let $\mathcal{A}$ be one of the algbras $\mathrm{L}^p_{\mathrm{cent}}(G)$ for $1 \le p < \infty$ or $\mathrm{C}_{\mathrm{cent}}(G)$ and define for an irreducible representation ξ of dimension d_ξ

$$h_\xi(f) = \frac{1}{d_\xi}\int_G f(x)\overline{\chi_\xi(x)}\mathrm{d}x.$$

Show that $h_\xi \in \operatorname{Hom}(\mathcal{A}, \mathbb{C}) \setminus \{0\} = \sigma(\mathcal{A})$ and that there is a bijection between equivalence classes of unitary representations $[\xi]$ and the spectrum $\sigma(\mathcal{A})$.

9.2. We consider the group $\mathrm{SU}(2)$ of unitary 2×2 matrices with determinant equal to 1 and one unitary representation $\boldsymbol{\pi} : \mathrm{SU}(2) \to \mathrm{U}(d)$ of dimension d.

(a) Let $T \subset \mathrm{SU}(2)$ be the set of diagonal matrices in $\mathrm{SU}(2)$, i.e., the set of matrices of the form

$$R_\theta = \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\theta} & \\ & \mathrm{e}^{-2\pi\mathrm{i}\theta} \end{pmatrix}, \qquad \theta \in \mathbb{R}$$

Show that there exist numbers $\alpha_k \in \mathbb{N}_0$ such that for all $\theta \in \mathbb{R}$

$$\operatorname{tr}\boldsymbol{\pi}(R_\theta) = \sum_{k\in\mathbb{Z}} \alpha_k \mathrm{e}^{2\pi\mathrm{i}k\theta}. \tag{*}$$

(b) Let $\theta_1, \theta_2 \in \mathbb{R}$. Show that there exists a matrix $A \in \mathrm{SU}(2)$ with

$$AR_{\theta_1} = R_{\theta_2}A$$

if and only if $\theta_1 + \theta_2 \in \mathbb{Z}$ or $\theta_1 - \theta_2 \in \mathbb{Z}$. Use this to prove that in (*) the relation $\alpha_k = \alpha_{-k}$ holds true for all k.

Topics as preparation

9.3. Schrödinger operators with periodic potential are operators of the form

$$A = -\Delta + q(x) : \mathrm{H}^2(\mathbb{R}^n) \to \mathrm{L}^2(\mathbb{R}^n)$$

where $q : \mathbb{R}^n \to \mathbb{R}$ is continuous and periodic with respect to some lattice $\Lambda \subset \mathbb{R}^n$

$$q(x + y) = q(x) \quad \text{for } x \in \mathbb{R}^n\ y \in \Lambda.$$

Let Ω be a translation cell of Λ and $\Xi = \mathbb{R}^n/\Lambda^\circ$.

Lemma. *The operator A decomposes as direct integral*

$$A = \int_{\Xi}^{\oplus} A(\xi)$$

for operators $A(\xi) : \mathrm{H}^2_\xi(\Omega) \to \mathrm{L}^2(\Omega)$, where $\mathrm{H}^2_\xi(\Omega)$ denotes the subspace of Sobolev functions $f \in \mathrm{H}^2(\Omega)$ satisfying ξ-quasiperiodic boundary conditions.

Lemma. *Each operator $A(\xi)$ has a compact resolvent $(\lambda - A(\xi))^{-1} \in \mathcal{L}(\mathrm{L}^2(\Omega))$.*

Lemma. *The spectrum of the operator A (seen as unbounded operator on $\mathrm{L}^2(\Omega)$) is purely essential spectrum and satisfies*

$$\sigma(A) = \bigcup_{\xi \in \Xi} \sigma_p(A(\xi)).$$