Harmonische Analysis — Blatt 8

Die Mehrheit bringt der Mathematik Gefühle entgegen, wie sie nach Aristoteles durch die Tragödie geweckt werden sollen, nämlich Mitleid und Furcht. Mitleid mit denen, die sich mit der Mathematik plagen müssen, und Furcht: daß man selbst einmal in diese gefährliche Lage geraten könne.
(Paul Epstein; 1883–1966)

Problems

8.1. Let $\boldsymbol{\mu}$ be a spectral measure defined on a measurable space Σ and taking values in a separable Hilbert space H. Define for some $u \in H$, $\|u\| = 1$,

$$\mathfrak{Z}(u) = \overline{\operatorname{span}}\{T_f u \mid f \in \mathrm{L}^\infty(\Sigma)\} \subset H$$

in terms of the measurable functional calculus.

Show that

$$\mathfrak{Z}(u) \simeq \mathrm{L}^2(\Sigma; \mathrm{d}\boldsymbol{\mu}_{u,u})$$

are isometrically-isomorphic.

Topics as preparation

8.2. The *spectral multiplicity theorem* characterises spectral measures up to unitary equivalence. In its original proof due to Hellinger and Hahn spectral measures are related to canonical models of operators and underlying Hilbert spaces constructed in terms of the spectral measure. Our aim is to reproduce this original proof.

Let H be saparable and $\boldsymbol{\mu}$ be an $\mathcal{L}(H)$-valued spectral measure on some measure space Σ. For elements $u, v \in H$ we write $u \ll v$ if the (positive probability) measures satisfy $\boldsymbol{\mu}_{u,u} \ll \boldsymbol{\mu}_{v,v}$, i.e., if all $\boldsymbol{\mu}_{v,v}$ null sets are also $\boldsymbol{\mu}_{u,u}$ null sets. A (possibly finite) sequence (u_k) is called a *Hellinger–Hahn sequence* if

$$u_{k+1} \ll u_k \qquad \text{for all } k = 1, 2, \ldots \tag{1}$$

and

$$H = \bigoplus_k \mathfrak{Z}(u_k), \qquad \text{where } \mathfrak{Z}(u) = \overline{\operatorname{span}}\{T_f u \mid f \in \mathrm{L}^\infty(\Sigma)\}, \tag{2}$$

hold true.

Theorem. *Let H be a separable Hilbert space and $\boldsymbol{\mu}$ be an $\mathcal{L}(H)$-valued spectral measure on some measure space Σ. Then there exists a Hellinger–Hahn sequence for $\boldsymbol{\mu}$.*

As with orthonormal bases, Hellinger–Hahn sequences are far from beeing unique. Nevertheless they are almost unique as the following theorem states.

Theorem. *Let (u_k) and (v_k) be two Hellinger–Hahn sequences for the same spectral measure. Then $u_k \ll v_k \ll u_k$ for all k.*

Let H_1 and H_2 be two separable Hilbert spaces. We call two spectral measures $\boldsymbol{\mu}$ (taking values in $\mathcal{L}(H_1)$) and $\boldsymbol{\nu}$ (taking values in $\mathcal{L}(H_2)$) on Σ unitarily equivalent if there is a unitary operator $U \in \mathcal{L}(H_1, H_2)$ such that $\boldsymbol{\mu}_{u,v} = \boldsymbol{\nu}_{Uu,Uv}$ for all $u, v \in H_1$.

Theorem (Hellinger–Hahn)**.** *The following two statements are equivalent*

(a) *The spectral measures $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are equivalent.*

(b) *For any Hellinger–Hahn sequence (u_k) of $\boldsymbol{\mu}$ and any Hellinger–Hahn sequence (v_k) of $\boldsymbol{\nu}$*

$$\boldsymbol{\mu}_{u_k,u_k} \ll \boldsymbol{\nu}_{v_k,v_k} \ll \boldsymbol{\mu}_{u_k,u_k}$$

holds true.

If we define the notion of a spectral multiplicity function $\mathfrak{m} : \Sigma \to \mathbb{N}_0 \cup \{\aleph_0\}$, then both spectral measures are equivalent if their spectral multiplicity functions coincide almost everywhere.