Harmonische Analysis — Blatt 7

Math is like Ophelia in Hamlet — charming and a bit mad.
(Alfred North Whitehead; 1861–1947)

Problems

7.1. **(a)** Show that a sequence $(c_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is of positive type on $\mathbb{Z}$ if and only if all matrices

$$\begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_N \\ c_{-1} & c_0 & c_1 & \cdots & c_{N-1} \\ c_{-2} & c_{-1} & c_0 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-N} & c_{1-N} & c_{2-N} & \cdots & c_0 \end{pmatrix}$$

are positive semi-definite.

Conclude that for any sequence of positive type $|c_n| \le c_0$ and $c_{-n} = \overline{c_n}$.

(b) Assume now that $(c_n)_{n\in\mathbb{Z}}$ is of positive type with $c_0 = 1$. Use Bochner's theorem to show that there exists a uniquely determined probability measure $\mu \in \mathbb{M}_+(\mathbb{T})$ with

$$c_n = \int_{\mathbb{T}} z^n \mathrm{d}\mu(z).$$

What is the cyclic representation associated to $(c_n)_{n\in\mathbb{Z}}$ in the Hilbert space $\mathrm{L}^2(\mathbb{T}, \mathrm{d}\mu)$?

(c) Prove that $(c_n)_{n\in\mathbb{Z}} \in \mathcal{P}_1(\mathbb{Z})$ is extremal if and only if $c_n = \omega^n$ for some $\omega \in \mathbb{T}$.

7.2. **(a)** Determine all characters of the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$.

(b) Find the associated Fourier transform for this group, determine its Plancherel identity and give the inversion formula for the transform.

Topics as preparation

7.3. Let p be a prime number. The p-adic numbers $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ with respect to the p-adic metric $|r-s|_p$ with[1]

$$|r|_p = p^{-a}, \qquad \text{where} \quad r = p^a \frac{m}{n}, \quad \gcd(n,p) = \gcd(m,p) = 1.$$

Let further $\mathbb{Z}_p$ denote the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$. It follows that elements of $\mathbb{Q}_p$ are given by (in $\mathbb{Q}_p$ convergent) series

$$x = \sum_{k=-N}^{\infty} a_k p^k, \qquad a_k \in \mathbb{Z}_{(p)} = \mathbb{Z}/(p\mathbb{Z})$$

with uniquely determined digits a_k.

(a) $\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is σ-compact.

(b) Characters of $\mathbb{Q}_p$ are of the form $\chi_y(x) = \chi(xy)$ with

$$\chi\left(\sum_{k=-N}^{\infty} a_k p^k\right) = \exp\left(2\pi\mathrm{i} \sum_{k=-N}^{-1} a_k p^k\right).$$

Furthermore, each $y \in \mathbb{Q}_p$ corresponds to a unique character χ_y and hence $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$.

[1] $\gcd(m,n)$ stands for the *greatest common divisor* of two integers m and n.

(c) If the Haar measure on $\mathbb{Q}_p$ is normalised in such a way that $|\mathbb{Z}_p| = 1$, then

$$\int_{\mathbb{Z}_p} \chi(xy)\mathrm{d}x = \begin{cases} 1, & y \in \mathbb{Z}_p \\ 0, & \text{otherwise.} \end{cases}$$

(d) Let $f \in \mathrm{L}^1(\mathbb{Q}_p)$. Denoting by

$$\widehat{f}(y) = \int_{\mathbb{Q}_p} f(x)\chi(-xy)\mathrm{d}y$$

the Fourier transform, then for any $f \in \mathrm{L}^1(\mathbb{Q}_p) \cap \mathrm{L}^2(\mathrm{Q}_p)$

$$\int_{\mathrm{Q}_p} |f(x)|^2 \mathrm{d}x = \int_{\mathrm{Q}_p} |\widehat{f}(y)|^2 \mathrm{d}y.$$

holds true.

<u>Source:</u> Gerald B. Folland *A Course in Abstract Harmonic Analysis*, Chapter 4.1