Harmonische Analysis — Blatt 6

Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy.
(Roger Bacon; 1214–1294)

Problems

6.1. **(a)** Let G be locally compact and abelian. Let further $\boldsymbol{\pi} : G \to \mathcal{L}(H)$ be an irreducible unitary representation of G. Show that $\dim H = 1$.

Notation: Using the identification $\mathrm{U}(1) = \mathbb{T} \subset \mathbb{C}$, irreducible unitary representations of locally compact abelian groups are nothing else than continuous functions $\chi : G \to \mathbb{C}$ with $|\chi(x)| = 1$ and satisfying $\chi(xy) = \chi(x)\chi(y)$. Such functions are called *characters* of G.

(b) Find all characters of the cyclic group $\mathrm{C}_n = \{\zeta \in \mathbb{C} \ : \ \zeta^n = 1\}$ and of the group $\mathbb{T} \subset \mathbb{C}$.

(c) Let $G = \{0,1\}^{\mathbb{N}}$ be the *Cantor group*. Show that the characters of G are given by the functions

$$\chi_\omega((\alpha_n)_{n\in\mathbb{N}}) = \prod_{j\in\omega} (-1)^{\alpha_j}$$

parameterised by *finite* subsets $\omega \subset \mathbb{N}$.

6.2. Consider again the Cantor group $G = \{0,1\}^{\mathbb{N}}$ together with the map

$$x : G \ni (\alpha_n)_{n\in\mathbb{N}} \to \sum_{n\in\mathbb{N}} \alpha_n 2^{-n} \in [0,1].$$

(a) Show that the Haar measure on G is mapped to the Lebesgue measure on $[0,1]$.

(b) Let $W_\omega : [0,1] \to \mathbb{C}$ be defined such that $W_\omega(x(\alpha)) = \chi_\omega(\alpha)$ is valid almost everywhere. Sow that the functions W_ω form an orthonormal basis of $\mathrm{L}^2[0,1]$.

Notation: The functions W_ω are called Walsh functions.

(c) We use the map x to define an almost everywhere defined addition $\boxplus$ on $[0,1]$ by requiring

$$x(\alpha) \boxplus x(\beta) = x(\alpha\beta) \qquad \text{a.e.}$$

Then for $f \in \mathrm{L}^2[0,1]$ and $y \in [0,1]$ we define the operator $T_y f(x) = f(x \boxplus y)$ and similarly for $\omega \subset \mathbb{N}$ the operator $M_\omega f(x) = W_\omega(x) f(x)$. Show that both are unitary and satisfy

$$T_{y_1} T_{y_2} = T_{y_1 \boxplus y_2}, \qquad M_{\omega_1} M_{\omega_2} = M_{\omega_1 \triangle \omega_2}, \qquad T_y M_\omega = W_\omega(y) M_\omega T_y$$

with the symmetric difference $\omega_1 \triangle \omega_2 = (\omega_1 \setminus \omega_2) \cup (\omega_2 \setminus \omega_1)$.

Topics as preparation

6.3. Construction of Haar measures for the groups

(a) $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) : \det A = 1\}$, the special linear group in dimension n;

(b) $\mathrm{SO}(3) = \{A \in \mathrm{SL}(3,\mathbb{R}) : A^\top A = I = AA^\top\}$, the group of special orthogonal matrices.

Source: Chapter IV.15 of Edwin Hewitt and Kenneth A. Ross *Abstract Harmonic Analysis*, Vol. 1. Springer 1963 for generic examples of Haar measures on semi-direct products of matrix groups